Particle Trajectory and Capture Radius around Multiple Wires on HGMS

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多線条大勾配磁気分離における粒子航跡と捕獲半径について

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Has a long history the technique collecting iron-components with magnetic field. It is also possible to gather paramagnetic materials in high gradient magnetic field. In order to obtain a wide capture region in high field-high gradient magnetic field, are recently utilized magnetic fine wires in high intensity field. We call it a device of HGMS(High Gradient Magnetic Separation or Separator).

Working devices of HGMS are, therefore, constructed with a great number of arranged magnetic wires. The analysis of particle trajectory in the capture region has been mainly considered, however, on a single capturing wire element. In this paper are shown the particle loci around multiple wires arranged in a line and the capture radius.

I. Introduction

The particle trajectory investigated so far at a single wire[1)2)] is extended to that of multiple capturing elements. That is why there are many wires in practical appratus for the purpose increasing the capture efficiency. It is necessary, therefore, for actual devices to consider the particle trajectory around multiple wires.

The capturing elements are arranged in a line, and external magnetic field are applied to parallel or perpendicular for the wire arrangement. And also for the two cases are considered the direction of flow.

The capture radius are given on the strength of the obtained trajectory equations and the comparison is done for the cases of a single and multiple wires.

II. Preliminary consideration for the case of a single wire (Observation on the influence by the value of k)[3)]

A circular cylindrical wire of radius a is placed in a homogenious magnetic field of strength H_0 (Fig.1).

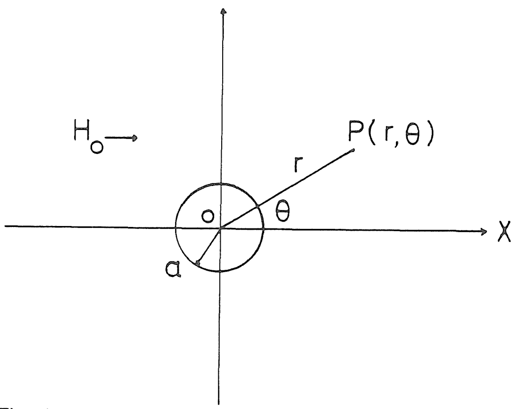

Fig. 1. Coordinates system used to calculate the field with a wire of radius a.

It follows from magnetostatics that magnetization by the magnetic field causes the cylinder to act as a two-dimentional dipole. The magnetic potential φ_m at a point $P(r, \theta)$ is given by

$$\varphi_m = -r H_0 \cos\theta + \frac{M_s a^2}{2\mu_0 r}\cos\theta \qquad \text{for} \qquad 2\mu_0 H_0 > M_s \tag{1}$$

$$= -r H_0 \cos\theta + \frac{\mu-\mu_0}{\mu+\mu_0}\frac{a^2}{r}\cos\theta \qquad \text{for} \qquad 2\mu_0 H_0 \leq M_s \tag{2}$$

where μ and M_s are the permeability and the saturation magnetization of the wire respectively.

For convienience the formula (1) is treated, and in the case of the condition $2\mu_0 H_0 \leq M_s$ the formula (2) may be used. The magnetic field H is given by

$$\mathbf{H} = -\text{grad}\,\varphi_m \tag{3}$$

That is, the components are given by

$$H_r = H_0 \cos\theta + \frac{M_s a^2}{2\mu_0 r^2}\cos\theta \tag{4}$$

$$H_\theta = -H_0 \sin\theta + \frac{M_s a^2}{2\mu_0 r^2}\sin\theta \tag{5}$$

The force acting a magnetic sphere with radius b at the point P is given by

$$\mathbf{F}_m = \frac{1}{2}\mu_0 x_s V_b \,\text{grad}(\mathbf{H}^2) \tag{6}$$

where x_s and V_p correspond to the relative magnetic susceptibility and the volume of the sphere respectively. Then the components are given by

$$F_{mr} = -\frac{4\pi x_s H_0 M_s a^2 b^3}{3}\left(\frac{M_s a^2}{2\mu_0 H_0 r^5} + \frac{\cos 2\theta}{r^3}\right) \tag{7}$$

$$F_{m\theta} = -\frac{4\pi x_s H_0 M_s a^2 b^3}{3}\cdot\frac{\sin 2\theta}{r^3} \tag{8}$$

In the case that a fluid velocity is small and the Reynolz number is low, the drag force F_D acting the sphere by the fluid is expressed by the following Stoke's formula.

$$\mathbf{F}_D = -6\pi\eta b \mathbf{v} \tag{9}$$

where η and $\mathbf{v}$ correspond to the viscosity of fluid and the relative velocity between the sphere and fluid.

Then, the equation of motion is given by

$$\mathbf{F}_m + \mathbf{F}_D = 0 \tag{10}$$

That is, the velocity components are given by

$$\frac{dr_a}{dt} = -\frac{v_m}{a}\left(\frac{k}{r_a^5} + \frac{\cos 2\theta}{r_a^3}\right) \tag{11}$$

$$r_a \frac{d\theta}{dt} = -\frac{v_m}{a}\frac{\sin 2\theta}{r_a^3} \tag{12}$$

where $r_a = \dfrac{r}{a}$, $\dfrac{v_m}{a} = \dfrac{2x_s H_0 M_s b^2}{9\eta a^2}$ and $k = \dfrac{M_s}{2\mu_0 H_0}$

From the formulas (11) and (12), the following differential equation is gotten.

$$\frac{dr_a}{d\theta} = \frac{1}{\sin 2\theta}\left(\frac{k}{r_a} + r_a \cos 2\theta\right) \tag{13}$$

And the solution is given by (See Annexe 1)

$$r_a = (c \sin 2\theta - k \cos 2\theta)^{\frac{1}{2}} \tag{14}$$

where $\quad c = \dfrac{(r_0^2 + k\cos 2\theta_0)}{\sin 2\theta_0}$

The coordinates of a initial position P_0 of the particle are r_0 and θ_0. The curves are shown in Fig. 2 with

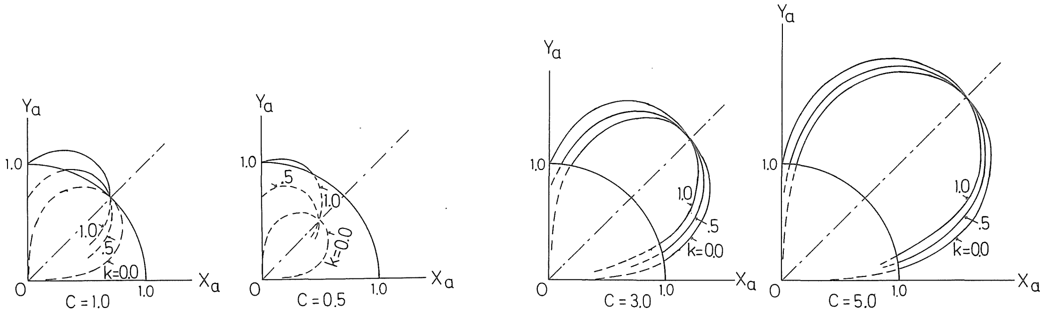

Fig. 2.　Variation of particle loci by k in case of flowless.

the condition $0 \leq k \leq 1$. These figures represent the influence of the value of k on the locus of the sphere. The dashed-curves written inside the wire are not of actual and the dot-dashed line is a bisector of x-y axis.

Ⅲ. Trajectory of particle and capture radius[7)]

1. Stream function

1.1 Consideration to the case that the wire arrangement is perpendicular to the stream

As shown in Fig. 3, the cylindrical magnetic wires with radius a are arranged along y-axis with equal distances ℓ and the fluid folws for x-axis direction with speed $-v_0$.

The stream potential function is given as follows. (Annexe 2)

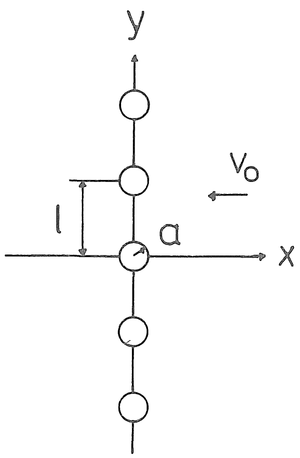

Fig. 3.　Wire arrangement and flow direction.

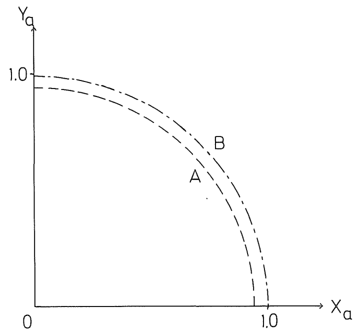

Fig. 4.　Deformation of wire.

$$\omega = \Phi + i\Psi = -v_0\left[z + \frac{a^2\pi}{\ell}\coth\left(\frac{\pi}{\ell}z\right)\right] \qquad (15)$$

Therefore

$$\Psi_a = -v_0\left[y_a - \frac{1}{2}\beta\frac{\sin\beta y_a}{(\mathrm{ch}\beta x_a - \cos\beta y_a)}\right] \qquad (16)$$

where $x_a = \frac{x}{a}$, $y_a = \frac{y}{a}$, $\beta = \frac{2\pi a}{\ell}$ and $\Psi_a = \frac{\Psi}{a}$.

The subscript a indicates the normalization by a.

The boundary surface is given by $\Psi_a = 0$, and we shall consider the surface configuration under the influence of neighboring wires is considered. The dashed curve A in Fig. 4 shows the surface of $\Psi_a = 0$ when $\ell/a = 5$.

For the purpose of keeping $r_a \simeq 1$, is introduced a correction factor c_1 at the second term in the braket of Eq. (16).

Then the following is gotten when $\Psi_a = 0$.

$$y_a = \frac{1}{2}\beta c_1 \frac{\sin\beta y_a}{(\mathrm{ch}\beta x_a - \cos\beta y_a)} \tag{17}$$

That is,

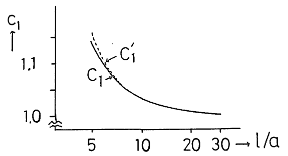

Fig. 5. Value of c_1 dy ℓ/a.

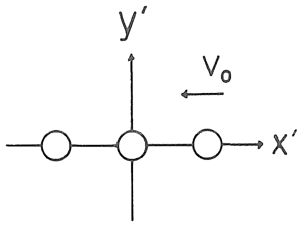

Fig. 6. Wire arrangement and flow direction.

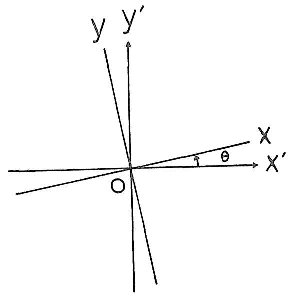

Fig. 7. Rotation of coordinates.

$$c_1 = \frac{2y_a(\mathrm{ch}\beta x_a - \cos\beta y_a)}{\beta\sin\beta y_a} \tag{18}$$

Putting $y_a = 0$, then

$$c_1 = \lim_{y_a \to 0} \frac{2y_a(\mathrm{ch}\beta x_a - 1)}{\beta \sin\beta y_a} = \left(\frac{2}{\beta}\mathrm{sh}\frac{\beta}{2}x_a\right)^2\Bigg|_{x_a=1} = \left(\frac{\ell}{\pi a}\mathrm{sh}\frac{\pi a}{\ell}\right)^2 \tag{19}$$

A solid line of Fig. 5 shows the value of c_1 with the variable ℓ/a, and we may regard that c_1 is nearly equal to 1 for $\ell/a \geq 10$.

And also for $x_a = 0$, the following is taken.

$$c_1' = \lim_{x_a \to 0} \frac{2y_a(\mathrm{ch}\beta x_a - \cos\beta y_a)}{\beta\sin\beta y_a} = \frac{2y_a(1-\cos\beta y_a)}{\beta\sin\beta y_a} = \frac{4y_a}{\beta\,2}\tan\frac{\beta y_a}{2}\Bigg|_{y_a=1} = \frac{2}{\beta}\tan\frac{\beta}{2} = \frac{(\ell/a)}{\pi}\tan\frac{\pi}{(\ell/a)} \tag{19}$$

A dashed line of the same figure shows c_1'. A attractive region for particles due to magnetic field is the neighbourhood at $x_a = 0$, and the value for c_1 may be taken that of the solid line in Fig. 5.

For $\ell/a = 5$, $c_1 = 1.14$ is adopted and the dot-dashed curve B in Fig. 4 corresponds to this case.

Then, the following stream function is obtained.

$$\Psi_a = -v_0 y_a\left[1 - \frac{1}{2}\beta^2 c_1 \frac{\sin\beta y_a/\beta y_a}{(\mathrm{ch}\beta x_a - \cos\beta y_a)}\right] \tag{20}$$

Accordingly, the velocity components v_{fx_1} and v_{fy_1} are

$$v_{fx_1} = \frac{\partial\Psi_a}{\partial y_a} = -v_0\left[1 + \frac{1}{2}\beta^2 c_1 \frac{1 - \mathrm{ch}\beta x_a \cdot \cos\beta y_a}{(\mathrm{ch}\beta x_a - \cos\beta y_a)^2}\right] \tag{21}$$

$$v_{fy_1} = -\frac{\partial\Psi_a}{\partial x_a} = v_0\left[\frac{1}{2}\beta^2 c_1 \frac{\mathrm{sh}\beta x_a \cdot \sin\beta y_a}{(\mathrm{ch}\beta x_a - \cos\beta y_a)^2}\right] \tag{22}$$

1.2 Consideration to the case that the wire arrangement is in parallel with the stream

Now, consider the occasion shown in Fig. 6.

The potential function is as follows. (Annexe 3)

$$\omega' = \Phi' + i\Psi' = -v_0\left[z' + \frac{a^2\pi}{\ell}\cot\frac{\pi}{\ell}z'\right] \tag{23}$$

Therefore, the stream function normalized by a Ψ_a' is given by

$$\Psi_a' = -v_0\left[1 - \frac{1}{2}\beta^2 \frac{\mathrm{sh}\beta y_a'/\beta y_a'}{(\mathrm{ch}\beta y_a' - \cos\beta x_a')}\right] \tag{24}$$

And then, consider a revolution of coordinates axis as indicated in Fig. 7. The formula of the revolving transformation of coordinates is shown as follows.

$$x' = x\cos\theta - y\sin\theta \tag{25}$$

$$y' = x\sin\theta + y\cos\theta$$

Accordingly, putting $\theta=\pi/2$ and applying to the expression (24), the following is gotten.

$$\Psi_a=-v_0 x_a\left[1-\frac{1}{2}\beta^2\frac{\mathrm{sh}\beta x_a/\beta x_a}{(\mathrm{ch}\beta x_a-\cos\beta y_a)}\right] \tag{26}$$

And also introduce a correction factor c_2 with the same manner as done in the last paragraph. In this case $c_2=0.88$ for $\ell/a=5$ and $c_2\simeq 1$ for $\ell/a\geq 10$.

Then, the velocity components v_{fx_2} and v_{fy_2} are

$$v_{fx_2}=\frac{\partial\Psi_a}{\partial y_a}=v_0\left[\frac{1}{2}\beta^2 c_2\frac{\mathrm{sh}\beta x_a\cdot\sin\beta y_a}{(\mathrm{ch}\beta x_a-\cos\beta y_a)^2}\right] \tag{27}$$

$$v_{fy_2}=-\frac{\partial\Psi_a}{\partial x_a}=-v_0\left[1-\frac{1}{2}\beta^2 c_2\frac{1-\mathrm{ch}\beta x_a\cdot\cos\beta y_a}{(\mathrm{ch}\beta x_a-\cos\beta y_a)^2}\right] \tag{28}$$

That is, the velocity in the case shown in Fig. 8 is gotten.

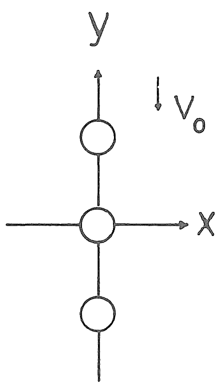

Fig. 8. Wire arrangement and flow direction.

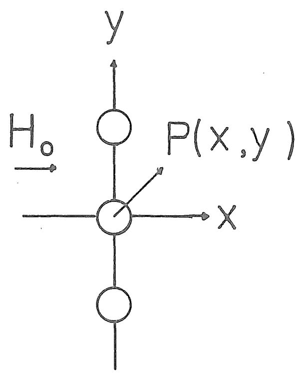

Fig. 9. Wire arrangement and external field direction.

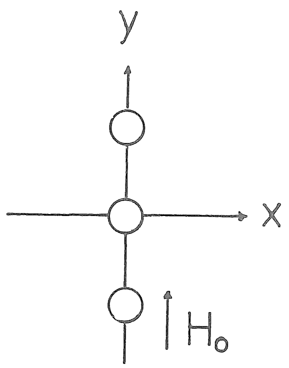

Fig. 10. Wire arrangement and external field direction.

2. Magnetic field of linearly arranged magnetic wires

2.1 Consideration to the case that the wire arrangement is perpendicular to the external field

Let the magnetic field be applied as shown in Fig. 9 and the magnetic wire be saturated.

The magnetic potential φ_m at the point $\mathrm{P}(x, y)$ is given by (Annexe 4)

$$\varphi_m(x, y)=-x\mathrm{H}_0+\frac{a^2\mathrm{M}_s}{2\mu_0}\sum_{m=-\infty}^{\infty}\frac{x}{x^2+(y-m\ell)^2}=-x\mathrm{H}_0+\frac{a^2\mathrm{M}_s}{2\mu_0}\cdot\frac{\pi}{\ell}\cdot\frac{\mathrm{sh}2\pi x/\ell}{(\mathrm{ch}2\pi x/\ell-\cos 2\pi y/\ell)} \tag{29}$$

Consequently,

$$\mathrm{H}_{x_1}=-\frac{\partial\varphi_m}{\partial x}=\mathrm{H}_0-\frac{1}{2}\frac{\mathrm{M}_s}{2\mu_0}\beta^2\frac{1-\mathrm{ch}\beta x_a\cdot\cos\beta y_a}{(\mathrm{ch}\beta x_a-\cos\beta y_a)^2} \tag{30}$$

$$\mathrm{H}_{y_1}=-\frac{\partial\varphi_m}{\partial y}=\frac{1}{2}\frac{\mathrm{M}_s}{2\mu_0}\beta^2\frac{\mathrm{sh}\beta x_a\cdot\sin\beta y_a}{(\mathrm{ch}\beta x_a-\cos\beta y_a)^2} \tag{31}$$

And the square sum is

$$\mathrm{H}_1{}^2=\mathrm{H}_{x_1}{}^2+\mathrm{H}_{y_1}{}^2=\mathrm{H}_0{}^2-\frac{\mathrm{M}_s}{2\mu_0}\beta^2\mathrm{H}_0\frac{1-\mathrm{ch}\beta x_a\cdot\cos\beta y_a-k\cdot\beta^2}{(\mathrm{ch}\beta x_a-\cos\beta y_a)^2} \tag{32}$$

Therefore the force acted on a magnetic particle with the volume V_p and the relative permeability X_S is given by

$$\mathrm{F}_{mx_1}=\frac{1}{2}\mathrm{V}_p\mu_0 X_S\frac{\partial\mathrm{H}_1{}^2}{\partial x}=-\frac{2\mathrm{V}_p X_S\mathrm{M}_s\pi^3 a^2\mathrm{H}_0}{\ell^2}\frac{(\mathrm{ch}\beta x_a+\cos\beta y_a)\cos\beta y_a-(2-1/2\cdot k\beta^2)}{(\mathrm{ch}\beta x_a-\cos\beta y_a)^3}\mathrm{sh}\beta x_a \tag{33}$$

$$\mathrm{F}_{my_1}=\frac{1}{2}\mathrm{V}_p\mu_0 X_S\frac{\partial\mathrm{H}_1{}^2}{\partial y}=-\frac{2\mathrm{V}_p X_S\mathrm{M}_s\pi^3 a^2\mathrm{H}_0}{\ell^3}\frac{(\mathrm{ch}\beta x_a+\cos\beta y_a)\mathrm{ch}\beta x_a-(2-1/2\cdot k\beta^2)}{(\mathrm{ch}\beta x_a-\cos\beta y_a)^3}\sin\beta y_a \tag{34}$$

2.2 Consideration to the case that the wire arrangement is in aprallel with the field

The potential shown in Fig. 10 is given by (Annexe 5)

$$\varphi_{m_2}(x,\ y)=-y\mathrm{H}_0+\frac{a^2\mathrm{M}_s}{2\mu_0}\sum_{m=-\infty}^{\infty}\frac{y-m\ell}{x^2+(y-m\ell)^2}=-y\mathrm{H}_0+\frac{a^2\mathrm{M}_s}{2\mu_0}\frac{\pi}{\ell}\frac{\sin\beta y_a}{(\mathrm{ch}\beta x_a-\cos\beta y_a)} \tag{35}$$

Consequently,

$$\mathrm{H}_{x_2}=-\frac{\partial\varphi_{m_2}}{\partial x}=\frac{1}{2}k\beta^2\frac{\mathrm{sh}\beta x_a\cdot\sin\beta y_a}{(\mathrm{ch}\beta x_a-\cos\beta y_a)^2} \tag{36}$$

$$\mathrm{H}_{y_2}=-\frac{\partial\varphi_{m_2}}{\partial y}=\mathrm{H}_0-\frac{1}{2}k\beta^2\frac{\mathrm{ch}\beta x_a\cdot\cos\beta y_a-1}{(\mathrm{ch}\beta x_a-\cos\beta y_a)^2} \tag{37}$$

And also

$$\mathrm{H}_2{}^2=\mathrm{H}_{x_2}{}^2+\mathrm{H}_{y_2}{}^2=\mathrm{H}_0{}^2+\frac{\mathrm{M}_s}{2\mu_0}\beta^2\mathrm{H}_0\frac{1-\mathrm{ch}\beta x_a\cdot\cos\beta y_a-1/4\cdot k\beta^2}{(\mathrm{ch}\beta x_a-\cos\beta y_a)^2} \tag{38}$$

Accordingly, the force components in this case are given by

$$\mathrm{F}_{mx_2}=\frac{1}{2}\mathrm{V}_p\mu_0X_s\frac{\partial\mathrm{H}_2^2}{\partial x}=\frac{4\pi\mathrm{X}_s\mathrm{M}_s\mathrm{H}_0b^3}{3}\frac{1}{4a}\beta^3\frac{(\mathrm{ch}\beta x_a+\cos\beta y_a)\cos\beta y_a-2+1/2\cdot k\beta^2}{(\mathrm{ch}\beta x_a-\cos\beta y_a)^3}\mathrm{sh}\beta x_a \tag{39}$$

$$\mathrm{F}_{my_2}=\frac{1}{2}\mathrm{V}_p\mu_0X_s\frac{\partial\mathrm{H}_2^2}{\partial y}=\frac{4\pi\mathrm{X}_s\mathrm{M}_s\mathrm{H}_0b^3}{3}\frac{1}{4a}\beta^3\frac{(\mathrm{ch}\beta x_a+\cos\beta y_a)\mathrm{ch}\beta x_a-2+1/2\cdot k\beta^2}{(\mathrm{ch}\beta x_a-\cos\beta y_a)^3}\sin\beta y_a \tag{40}$$

3. Equation of motion for a particle

Let the stream speed and the particle velocity v_f and v_p respectively. The drag force F_D acting on a particle with the radius b is

$$\mathrm{F}_\mathrm{D}=6\pi\eta b(\mathrm{v}_f-\mathrm{v}_p) \tag{41}$$

Now, the motion of particle is decided by the following relation.

$$\mathrm{F}_\mathrm{D}+\mathrm{F}_m=0 \tag{42}$$

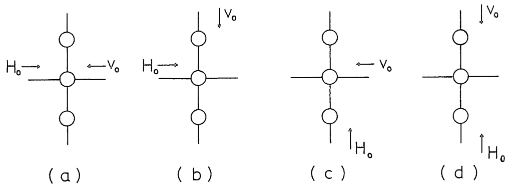

Fig. 11. Relation among the wire arrangement, the flow direction and the external field.

And, apply the equation of motion to the case shown in Fig. 11.

Case (a)

The components are given by

$$\frac{dx}{dt}=v_{fx_1}+\frac{\mathrm{F}_{mx_1}}{6\pi\eta b} \tag{43}$$

$$\frac{dy}{dt}=v_{fy_1}+\frac{\mathrm{F}_{my_1}}{6\pi\eta b} \tag{44}$$

Accordingly,

$$\frac{dy}{dx}=-\frac{\frac{1}{2}\beta^2c_1\frac{\mathrm{sh}\beta x_a\cdot\sin\beta y_a}{(\mathrm{ch}\beta x_a-\cos\beta y_a)^2}-\frac{1}{4}\beta^3\left(\frac{v_m}{v_0}\right)\frac{(\mathrm{ch}\beta x_a+\cos\beta y_a)\mathrm{ch}\beta x_a-(2-1/2\cdot k\beta^2)}{(\mathrm{ch}\beta x_a-\cos\beta y_a)^3}\sin\beta y_a}{1+\frac{1}{2}\beta^2c_1\frac{1-\mathrm{ch}\beta x_a\cdot\cos\beta y_a}{(\mathrm{ch}\beta x_a-\cos\beta y_a)}+\frac{1}{4}\beta^3\left(\frac{v_m}{v_0}\right)\frac{(\mathrm{ch}\beta x_a+\cos\beta y_a)\cos\beta y_a-(2-1/2\cdot k\beta^2)}{(\mathrm{ch}\beta x_a-\cos\beta y_a)^3}\mathrm{sh}\beta x_a} \tag{45}$$

where $v_m=\frac{2X_s\mathrm{M}_s\mathrm{H}_0b^2}{9\eta a}$ is called "a magnetic velocity". If $k=0$ (this assumption may be comparatively good as shown in the preliminary consideration and the influence on capture radius by k is considered elsewhere[8]), the following solution is obtained.

$$y_a-\frac{1}{2}\beta c_1\frac{\sin\beta y_a}{(\mathrm{ch}\beta x_a-\cos\beta y_a)}+\frac{1}{4}\alpha\beta^2\frac{\mathrm{sh}\beta x_a\cdot\sin\beta y_a}{(\mathrm{ch}\beta x_a-\cos\beta y_a)^2}=\mathrm{const.} \tag{46}$$

where $\alpha=\frac{v_m}{v_0}$ and $\beta=\frac{2\pi a}{\ell}$

Case (b)

Similarly,

$$\frac{dx}{dt}=v_{fx_2}+\frac{F_{mx_1}}{6\pi\eta b} \tag{47}$$

$$\frac{dy}{dt}=v_{fy_2}+\frac{F_{my_1}}{6\pi\eta b} \tag{48}$$

Then,

$$\frac{dy}{dx}=-\frac{1+\frac{1}{2}\beta^2 c_2\frac{\mathrm{ch}\beta x_a\cdot\cos\beta y_a-1}{(\mathrm{ch}\beta x_a-\cos\beta y_a)^2}+\frac{1}{4}\beta^3\left(\frac{v_m}{v_0}\right)\frac{(\mathrm{ch}\beta x_a+\cos\beta y_a)\mathrm{ch}\beta x_a-(2-1/2\cdot k\beta^2)}{(\mathrm{ch}\beta x_a-\cos\beta y_a)^3}\sin\beta y_a}{\frac{1}{2}\beta^2 c_2\frac{\mathrm{sh}\beta x_a\cdot\sin\beta y_a}{(\mathrm{ch}\beta x_a-\cos\beta y_a)^2}-\frac{1}{4}\beta^3\left(\frac{v_m}{v_0}\right)\frac{(\mathrm{ch}\beta x_a+\cos\beta y_a)\cos\beta y_a-(2-1/2\cdot k\beta^2)}{(\mathrm{ch}\beta x_a-\cos\beta y_a)^3}\mathrm{sh}\beta x_a} \tag{49}$$

Likely, when $k=0$, the solution is given by

$$x_a-\frac{1}{2}\beta c_2\frac{\mathrm{sh}\beta x_a}{(\mathrm{ch}\beta x_a-\cos\beta y_a)}-\frac{1}{4}\alpha\beta^2\frac{\mathrm{sh}\beta x_a\cdot\sin\beta y_a}{(\mathrm{ch}\beta x_a-\cos\beta y_a)^2}=\mathrm{const} \tag{50}$$

Case (c)

$$\frac{dx}{dt}=v_{fx_1}+\frac{F_{mx_2}}{6\pi\eta b} \tag{51}$$

$$\frac{dy}{dt}=v_{fy_1}+\frac{F_{my_2}}{6\pi\eta b} \tag{52}$$

Then

$$\frac{dy}{dx}=-\frac{\frac{1}{2}\beta^2 c_1\frac{\mathrm{sh}\beta x_a\cdot\sin\beta y_a}{(\mathrm{ch}\beta x_a-\cos\beta y_a)^2}+\frac{1}{4}\beta^3\left(\frac{v_m}{v_0}\right)\frac{(\mathrm{ch}\beta x_a+\cos\beta y_a)\mathrm{ch}\beta x_a-(2-1/2\cdot k\beta^2)}{(\mathrm{ch}\beta x_a-\cos\beta y_a)^3}\sin\beta y_a}{1+\frac{1}{2}\beta^2 c_1\frac{1-\mathrm{ch}\beta x_a\cdot\cos\beta y_a}{(\mathrm{ch}\beta x_a-\cos\beta y_a)^2}-\frac{1}{4}\beta^3\left(\frac{v_m}{v_0}\right)\frac{(\mathrm{ch}\beta x_a+\cos\beta y_a)\cos\beta y_a-(2-1/2\cdot k\beta^2)}{(\mathrm{ch}\beta x_a-\cos\beta y_a)^3}\mathrm{sh}\beta x_a} \tag{53}$$

Similarly, when $k=0$, the solution is following.

$$y_a-\frac{1}{2}\beta c_1\frac{\sin\beta y_a}{(\mathrm{ch}\beta x_a-\cos\beta y_a)}-\frac{1}{4}\alpha\beta^2\frac{\mathrm{sh}\beta x_a\cdot\sin\beta y_a}{(\mathrm{ch}\beta x_a-\cos\beta y_a)^2}=\mathrm{const.} \tag{54}$$

Case (d)

$$\frac{dx}{dt}=v_{fx_2}+\frac{F_{mx_2}}{6\pi\eta b} \tag{55}$$

$$\frac{dy}{dt}=v_{fy_2}+\frac{F_{my_2}}{6\pi\eta b} \tag{56}$$

Accordingly,

$$\frac{dy}{dx}=-\frac{1+\frac{1}{2}\beta^2 c_2\frac{\mathrm{ch}\beta x_a\cdot\cos\beta y_a-1}{(\mathrm{ch}\beta x_a-\cos\beta y_a)^2}-\frac{1}{4}\beta^3\left(\frac{v_m}{v_0}\right)\frac{(\mathrm{ch}\beta x_a+\cos\beta y_a)\mathrm{ch}\beta x_a-(2-1/2\cdot k\beta^2)}{(\mathrm{ch}\beta x_a-\cos\beta y_a)^3}\sin\beta y_a}{\frac{1}{2}\beta^2 c_2\frac{\mathrm{sh}\beta x_a\cdot\sin\beta y_a}{(\mathrm{ch}\beta x_a-\cos\beta y_a)}+\frac{1}{4}\beta^3\left(\frac{v_m}{v_0}\right)\frac{(\mathrm{ch}\beta x_a+\cos\beta y_a)\cos\beta y_a-(2-1/2\cdot k\beta^2)}{(\mathrm{ch}\beta x_a-\cos\beta y_a)^3}\mathrm{sh}\beta x_a} \tag{57}$$

Similarly, when $k=0$, the solution is gotten.

$$x_a-\frac{1}{2}\beta c_2\frac{\mathrm{sh}\beta x_a}{(\mathrm{ch}\beta x_a-\cos\beta y_a)}+\frac{1}{4}\alpha\beta^2\frac{\mathrm{sh}\beta x_a\cdot\sin\beta y_a}{(\mathrm{ch}\beta x_a-\cos\beta y_a)^2}=\mathrm{const.} \tag{58}$$

The results which are obtained in the method described above are summarized in Table Ⅰ.

4. Examples of particle trajectory

Some examples of locus obtained by the equations of motion derived in the preceeding paragraph are shown in Fig. 12 and Fig. 13. Fig. 14 shows that of the limiting case $\ell/a\to\infty$, that is, for a single wire.

5. Capture radius

The capture radius for each case is obtained by the particle trajectory equations. The results are given in

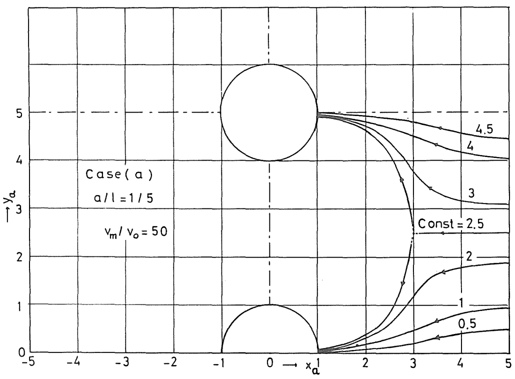

Fig. 12. Typical particle trajectory for the case (a) or (d).

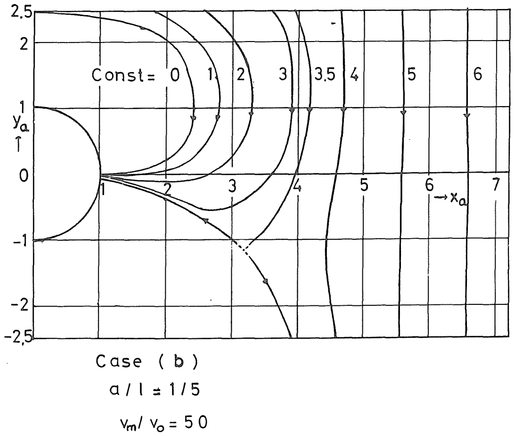

Fig. 13. Typical particle trajectory for the case (b) or (c).

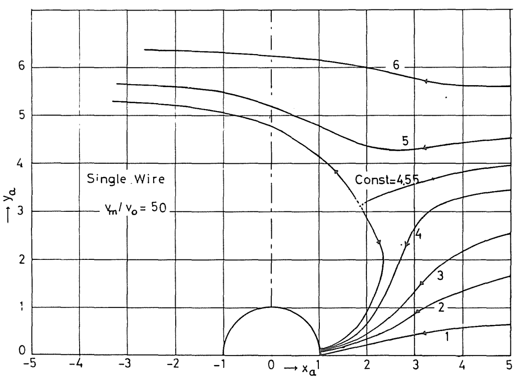

Fig. 14. For the case of a single wire.

Case	Arrangement	Equation of Motion	Equation of Motion for $\ell/a \to \infty$
(a)	H_o →, v_o ←	Formula (46)	$y_a - \frac{y_a}{x_a^2 + y_a^2} + (v_m/v_o) \times \frac{x_a y_a}{(x_a^2+y_a^2)^2} = \text{const.}$
(b)	H_o →, v_o ↓	Formula (50)	$x_a - \frac{x_a}{x_a^2 + y_a^2} - (v_m/v_o) \times \frac{x_a y_a}{(x_a^2+y_a^2)^2} = \text{const.}$
(c)	H_o ↑, v_o ←	Formula (54)	$y_a - \frac{y_a}{x_a^2 + y_a^2} - (v_m/v_o) \times \frac{x_a y_a}{(x_a^2+y_a^2)^2} = \text{const.}$
(d)	H_o ↑, v_o ↓	Formula (58)	$x_a - \frac{x_a}{x_a^2 + y_a^2} + (v_m/v_o) \times \frac{x_a y_a}{(x_a^2+y_a^2)^2} = \text{const.}$

Table I. Equation of Motion for each Case.

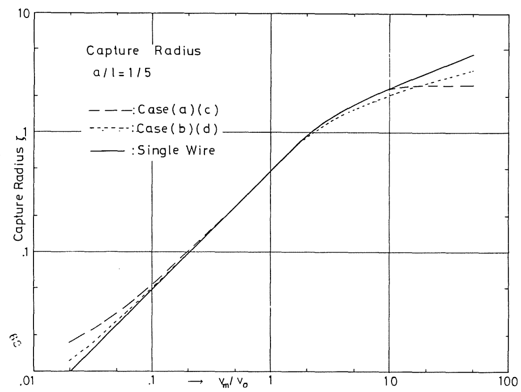

Fig. 15. Capture radius.

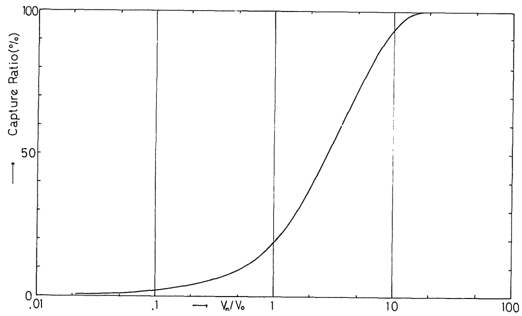

Fig. 16. Capture percentage for case (a).

Fig. 15. In the case (a) the circumstances that all particles are captured are come about. Therefore the capture ratio for particle is shown in Fig. 16.

Ⅳ. Conclusion and Acknowledgment

The expressions on trajectory among multiple wire arranged in a line are given on condition that $k = 0$ and the capture radius is obtained. For the influence of k over particle trajectory the tendency is examined also.

With the formula obtained will be possible the extension for the case of many parallel wires.

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Annexe 1

Solution of Eq. (13)

$$\frac{dr_a}{d\theta}=\frac{1}{\sin 2\theta}\left(\frac{k}{r_a}+r_a\cos 2\theta\right) \tag{13}$$

That is,

$$\frac{dr_a}{d\theta}-\cot 2\theta\cdot r_a=\frac{k}{r_a\sin 2\theta}$$

Putting $r_a=u$, then $2r_a\frac{dr_a}{d\theta}=\frac{du}{d\theta}$ And $2r_a\frac{dr_a}{d\theta}-2\cot 2\theta\cdot r_a^2=\frac{2k}{\sin 2\theta}$

Then, the solution is given by

$$u=e^{2\int \cot 2\theta d\theta}\left[\int\frac{2k}{\sin 2\theta}e^{-2\int \cot 2\theta d\theta}d\theta+\mathrm{c}\right]=e^{\ln|\sin 2\theta|}\left[-k\cot 2\theta+\mathrm{c}\right]$$

Therefore

$$r_a^2=|\sin 2\theta|\left[-k\cot 2\theta+\mathrm{c}\right]$$

Consider principally the region $0\leq\theta\leq\frac{\pi}{2}$, then

$$r_a^2=c\sin 2\theta-k\cos 2\theta \tag{1-1}$$

$$=c'\sin 2(\theta-\alpha) \tag{1-2}$$

where $c'=\sqrt{c^2+k^2}$, $\tan 2\alpha=k/c$

For a initial position we take $r=r_0$ and $\theta=\theta_0$, then

$$r_0^2=c\sin 2\theta_0-k\cos 2\theta_0 \tag{1-3}$$

therefore

$$c'=\sqrt{c^2+k^2}=\frac{\sqrt{r_0^4+2kr_0^2\cos 2\theta_0+k^2}}{\sin 2\theta_0}$$

$$\tan 2\alpha=k/c=k\sin 2\theta_0/(r_0^2+k\cos 2\theta_0)$$

And also, by

$$c=(r_0^2+k\cos 2\theta_0)/\sin 2\theta_0 \tag{1-4}$$

Applying (1-4) to (1-1), the following formula is gotten also[4].

$$r_a^2=\frac{r_0^2+k\cos 2\theta_0}{\sin 2\theta_0}\sin 2\theta-k\cos\theta=\frac{\sin 2\theta}{\sin 2\theta_0}r_0+k(\cot 2\theta_0\cdot\sin\theta-\cos 2\theta) \tag{1-5}$$

Annexe 2

The stream potential function for a single wire with radius a is, as known well, given by[5]

$$\omega=z+\frac{a^2}{z}$$

For the case of the arrangement of multiple wires lined along y-axis with interval ℓ, the expression is given by

$$\omega=z+\frac{a^2}{z}+\sum_{m=1}^{\infty}\left[\frac{a^2}{z-im\ell}+\frac{a^2}{z+im\ell}\right]=z+a^2\left[\frac{1}{z}+2z\sum_{m=1}^{\infty}\frac{1}{z^2+(m\ell)^2}\right]$$

Now, with $\coth z=\frac{1}{z}+2z\sum_{m=1}^{\infty}\frac{1}{z^2+(m\pi)^2}$

Accordingly,

$$\coth\left(\frac{\pi}{\ell}z\right)=\frac{1}{\left(\frac{\pi}{\ell}z\right)}+2\left(\frac{\pi}{\ell}z\right)\sum_{m=1}^{\infty}\frac{1}{\left(\frac{\pi}{\ell}z\right)^2+(m\pi)^2}=\frac{1}{\left(\frac{\pi}{\ell}\right)}\left[\frac{1}{z}+2z\sum_{m=1}^{\infty}\frac{1}{z^2+(m\ell)^2}\right]$$

Then the following expression is obtained.

$$\omega = z + \frac{a^2\pi}{\ell}\coth\left(\frac{\pi}{\ell}z\right)$$

Annexe 3

For the case arranged along x-axis, the expression is given by

$$\omega = z + \frac{a^2}{z} + \sum_{m=1}^{\infty}\left[\frac{a^2}{z-m\ell} + \frac{a^2}{z+m\ell}\right] = z + a^2\left[\frac{1}{z} + 2z\sum_{m=1}^{\infty}\frac{1}{z^2-(m\ell)^2}\right]$$

Now,

$$\cot z = \frac{1}{z} + 2z\sum_{m=1}^{\infty}\frac{1}{z^2-(m\pi)^2}$$

Therefore

$$\omega = z + \frac{a^2\pi}{\ell}\cot\left(\frac{\pi}{\ell}z\right)$$

Annexe 4

Calculation of $f(m) = \sum_{m=-\infty}^{\infty}\frac{x}{x^2+(y-m\ell)^2}$ [6)]

Representing its Fourier transformation by $\hat{f}(2\pi n)$, the following expression is obtained.

$$\hat{f}(2\pi n) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{x}{x^2+(y-t\ell)^2}e^{-i2\pi nt}dt$$

Putting $y - t\ell = -y_1$, $k = 2\pi n/\ell$

$$\hat{f}(2\pi n) = \frac{1}{\ell}e^{-iky}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{x}{x^2+{y_1}^2}e^{-iky_1}dy_1$$

While, for $x > 0$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{x^2+{y_1}^2}e^{-iky_1}dy_1 = \sqrt{\frac{\pi}{2}}\cdot\frac{e^{-x\cdot k}}{x} \qquad \text{for} \quad k > 0$$

$$= \sqrt{\frac{\pi}{2}}\cdot\frac{e^{x\cdot k}}{x} \qquad \text{for} \quad k < 0$$

For $x < 0$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{x^2+{y_1}^2}e^{-iky_1}dy_1 = \sqrt{\frac{\pi}{2}}\cdot\frac{e^{x\cdot k}}{-x} \qquad \text{for} \quad k > 0$$

$$= \sqrt{\frac{\pi}{2}}\cdot\frac{e^{-x\cdot k}}{-x} \qquad \text{for} \quad k < 0$$

Therefore, for $x > 0$

$$\sqrt{2\pi}\hat{f}(2\pi n) = \frac{\pi}{\ell}e^{-k(x+iy)} \qquad \text{for} \quad k > 0$$

$$= \frac{\pi}{\ell}e^{k(x-iy)} \qquad \text{for} \quad k < 0$$

For $x < 0$

$$\sqrt{2\pi}\hat{f}(2\pi n) = -\frac{\pi}{\ell}e^{k(x-iy)} \qquad \text{for} \quad k > 0$$

$$= -\frac{\pi}{\ell}e^{-k(x+iy)} \qquad \text{for} \quad k < 0$$

Accordingly, for $x > 0$

$$\sum_{m=-\infty}^{\infty}\frac{x}{x^2+(y-m\ell)^2} = \frac{\pi}{\ell}\sum_{n=0}^{\infty}e^{-\frac{2\pi n}{l}(x+iy)} + \frac{\pi}{\ell}\sum_{n=-1}^{-\infty}e^{\frac{2\pi n}{l}(x-iy)} = \frac{\pi}{\ell}\left[\frac{1}{1-e^{-\frac{2\pi}{l}(x+iy)}} + \frac{e^{-\frac{2\pi}{l}(x-iy)}}{1-e^{-\frac{2\pi}{l}(x-iy)}}\right]$$

$$= \frac{\pi}{\ell}\left[\frac{\mathrm{sh}2\pi x/\ell}{\mathrm{ch}2\pi x/\ell - \cos 2\pi y/\ell}\right]$$

And also, for $x<0$

$$\sum_{m=-\infty}^{\infty}\frac{x}{x^2+(y-m\ell)^2}=-\frac{\pi}{\ell}\left[\sum_{n=0}^{\infty}e^{\frac{2\pi n}{l}(x-iy)}+\sum_{n=-1}^{-\infty}e^{-\frac{2\pi n}{l}(x+iy)}\right]=\frac{\pi}{\ell}\left[\frac{\mathrm{sh}\,2\pi x/\ell}{\mathrm{ch}\,2\pi x/\ell-\cos 2\pi y/\ell}\right]$$

Annexe 5

Calculation of $f(m)=\sum_{m=-\infty}^{\infty}\frac{y-m\ell}{x^2+(y-m\ell)^2}$

Similarly, by the Fourier transformation

$$\hat{f}(2\pi n)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{y-t\ell}{x^2+(y-t\ell)^2}e^{-i2\pi nt}\,dt$$

Putting $y-t\ell=-y_1$, $k=2\pi n/\ell$

$$\hat{f}(2\pi n)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{y_1}{x^2+y_1{}^2}e^{-i\frac{2\pi}{l}n(y+y_1)}\frac{dy_1}{\ell}=\frac{1}{\ell}e^{-iky}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{y_1}{x^2+y_1{}^2}e^{-iky_1}\,dy_1$$

While, for $x>0$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{y_1}{x^2+y_1{}^2}e^{-iky_1}\,dy_1=-\mathrm{sign}(k)\sqrt{\frac{\pi}{2}}e^{-x|k|}$$

Therefore, for $x>0$

$$\sqrt{2\pi}\hat{f}(2\pi n)=-\frac{\pi}{\ell}e^{-k(x+iy)}\qquad \text{for}\quad k>0$$

$$=\frac{\pi}{\ell}e^{k(x-iy)}\qquad \text{for}\quad k<0$$

For $x<0$

$$\sqrt{2\pi}\hat{f}(2\pi n)=-\frac{\pi}{\ell}e^{k(x-iy)}\qquad \text{for}\quad k>0$$

$$=\frac{\pi}{\ell}e^{-k(x+iy)}\qquad \text{for}\quad k>0$$

On account of $\hat{f}(0)=0$ when $n=0$, the following is obtained.

For $x>0$

$$\sum_{m=-\infty}^{\infty}\frac{y-m\ell}{x^2+(y-m\ell)^2}=-\frac{\pi}{\ell}\sum_{n=1}^{\infty}e^{-\frac{2\pi n}{l}(x+iy)}+\frac{\pi}{\ell}\sum_{n=-1}^{-\infty}e^{\frac{2\pi n}{l}(x-iy)}=\frac{\pi}{\ell}\left[\frac{\sin 2\pi y/\ell}{\mathrm{ch}\,2\pi x/\ell-\cos 2\pi y/\ell}\right]$$

And also, for $x<0$

$$\sum_{m=-\infty}^{\infty}\frac{y-m\ell}{x^2+(y-m\ell)^2}=-\frac{\pi}{\ell}\sum_{n=1}^{\infty}e^{\frac{2\pi n}{l}(x-iy)}+\frac{\pi}{\ell}\sum_{n=-1}^{-\infty}e^{-\frac{2\pi n}{l}(x+iy)}=\frac{\pi}{\ell}\left[\frac{\sin 2\pi y/\ell}{\mathrm{ch}\,2\pi x/\ell-\cos 2\pi y/\ell}\right]$$

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