An Appropriate Method for The Maximum Capacity Route Problem [I]

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ネット・ワークの最大能力の最適方法

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The maximum capacity route problem is to find a route $R = \{(s, i), (i, j), \cdots, (k, m), (m, t)\}$ from the source s to the sink t wnose capacity $c(R) = \min_{(i,j)\in R} c_{ij}$ is maximum, where c_{ij} is the capacity of the directed edge (i, j). It is the route that allows the greatest flow from s to t.

Let $B = (S, F)$ be a subgraph of a given directed graph $G = (V, E)$ such that, for every $i \in S$, B contains a unique route from s to i, that is ,B is a directed tree rooted at s. We shall describe a simple labeling procedure for gradually expanding the directed tree rooted at s. Each time the tree is expanded, the new tree, B, will be assigned a value h(B). When the tree B reaches the sink t, B contains an unique route from s to t which is the maximum capacity route in G with the capacity h(B).

i) Start with $B = (\{s\}, \phi)$ and $h(B) = \infty$

ii) Given a tree $B = (S, F)$ with $t \in S$, from a new tree $B' = (S', F')$ as follows: First, find the value c such that,

(1) $$c = \max\{c_{rj} : r \in s \text{ and } j \in T\}$$

where $T = V - S$. Then, define a subset, D, of the cut (S, T) as

(2) $$D = \{(i, j) / (i, j) \in (S, T) \text{ and } c_{ij} \geq h\},$$

where

(3) $$h = \min\{h(B), c\}.$$

Also let

(4) $$K = \{k \in T / (i, k) \in D\}.$$

Now, for every $k \in K$, we select an edge $(i, k) \in D$ and add to F to from an new F'. And also

$$S' = S \cup K. \quad \text{Then, set}$$

(5) $$h(B') = h.$$

At this point, one may define a capacity transformation

(6) $$c'_{ik} = h \text{ for every } (i, k) \in D,$$

though it is not essential in our algorithm. We, thus, simply repeat this tree expansion until either the tree reaches the sink t, or the cut (S, T) for a tree $B = (S, F)$ is empty.

It is to be noted that, by the way of constructing the tree, $B = (S, F)$, and defining h(B), the route in the tree from s to any $i \in S$ has a capacity equal to or greater than h(B). In particular, when a new tree $B' = (S', F')$ is formed from $B = (S, F)$ by adding new vertices K, the route in the tree from s to any $k \in K$ is equal to h(B'). When the capacity transformation defined in (6) is also performed during procedure, the edge capacities along the route, $\{(s, i), (i, j) \cdots, (m, n)\}$, in the tree $B = (S, F)$ from s to any $n \in S$ from a non-increasing sequence, i.e., $c_{si} \gg c_{ij} \gg \cdots \gg c_{mn}$, and thus the route capacity is determind by the capacity of the last edge, (m, n), along the route. We now show,

Theorem: When the tree B reaches the sink t, the route in the tree from s to t is a maximum capacity route in G with the capacity h(B).

Proof: Since any route, R, from s to t and any cut, C, separating s and t in G, have at least an edge, (p, q), in common, i.e., $R \cap C \neq \phi$, we have

$$c(R) \ll c_{pp} \ll m(C), \text{ where}$$

(7) $$c(R) = \min\{c_{ij} : (i, j) \in R\} \text{ and}$$

(8) $$m(C) = \max\{c_{ij} : (i, j) \in C\}.$$

Hence, if we specify a procedure to find a route, R^*, such that $c(R^*) = m(C^*)$ for some cut, C^*, then R^* is proved to be a maximum capacity route.

Now, let B^* be the tree which reaches the sink t for the first time, then $h(B^*)$ is the capacity of the route R^* in the tree from s to t. However, by the way in which h(B) is determined by the procedure described in (1), (3) and (5), $h(B^*)$ represents the value $c = \max\{c_{rj} : r \in S,\ j \in T\}$ for some cut (S, T) with $s \in S$ and $t \in T$. This completes the proof.

If the procedure stops short of reaching t, the cut (S, T) for some tree B = (S, F) is empty and there is no route from s to t in G. Since the set S expands by at least one vertex at each iteration, the tree will necessarily reach t if there exists at least one route from s to t in G.

In fact, our algorithm provides a new proof to the min-max theorem concerning routes and cuts which was first pointed out by D. R. Fulkerson:

Theorem: let c(R) and m(C) be defined as in (7) and (8), then

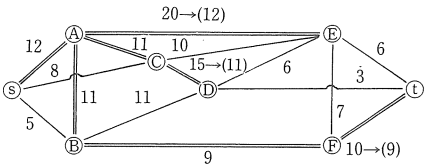
$$\max_{R \in \tilde{R}} c(R) = \min_{C \in \tilde{C}} m(C)$$

where $\tilde{R}$ is the collection of routes from s to t in G, and $\tilde{C}$ is the collection of cuts separating s and t in G.

As a variation of the algorithm, given a subgraph B = (S, F), one may from a new subgraph B′ = (S′, F′) by adding all edges of D defined in (2) to F. Then, the expanding subgraph may not be a directed tree and when the subgraph reaches t for the first time, any route in B from s to t is a maximum capacity route.

Example:

We shall find a maximum capacity route in the following mixed network. The transformed capacities will be indicated by the symbol→( ) along edges. The labeled edges show the tree B when it reached t for the first time:



Step 1: S = {s}; (S, T) = {(s, A), (s, C), (s, B)}
h = min{∞,12} = 12, K = {A}.

Step 2: S = {s, A}; (S,T) = {(s, B), (s, C), (A, B), (A, C), (A,E)},
h = min{12, 20} = 12, K = {E}.

Step 3: S = {s, A, E}; (S, T) = {(s, B), (s, C), (A, B), (A, C), (E, C), (E, D), (E, F), (E, t)},
h = min{12, 11} = 11, K = {C, B}.

Step 4: S = {s, A, B, C, E, }, (S, T) = {(B, F), (B, D), (C, D), (E, D), (E, F), (E, t)},
h = min{11, 15} = 11, K = {D}.

Step 5: S = {s, A, B, C, D, E}; (S, T) = {(B, F), (D, t), (E, F), (E, t)},
h = min{11, 9} = 9, K = {F}.

Step 6: S = {s, A, B, C, D, E, F}; (S, T) = {(D, t), (E, t), (F, t)}.
h = min{9, 10} = 9, K = {t}.
t ∈ S′, terminate. The terminal value O + h = 9.

The maximum capacity route = {(s, A), (A, B), (B, F), (F, t)} with the capacity 9.

Refernces

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