Business Administration

Studies on Extended Cumulative Damage Models and Their Applications to Garbage Collections

Xufeng Zhao

Ph.D. dissertation, February 2013

Aichi Institute of Technology
Toyota, Japan

Business Administration

Studies on Extended Cumulative Damage Models and Their Applications to Garbage Collections

Xufeng Zhao

Ph.D. dissertation, February 2013

Aichi Institute of Technology
Toyota, Japan

Studies on Extended Cumulative Damage Models and Their Applications to Garbage Collections

Prepared by: Xufeng Zhao

Supervisor: Professor Toshio Nakagawa

Release date:	February 2013
Category:	Public
Edition:	Final version
Keywords:	Reliability, maintenance, damage model, garbage collection
Comments:	This dissertation is submitted in partial fulfillment for the degree of Doctor of Philosophy at Aichi Institute of Technology

Department of Business Administration
Aichi Institute of Technology
Toyota, 470-0392, Japan

http://www.ait.ac.jp/index.html
Tel: (+81) 565-48-8121
Fax: (+81) 565-48-0509
E-mail: g09184gg@aitech.ac.jp

Abstract

This dissertation proposes several maintenance policies for extended cumulative damage models in reliability theory and their applications to garbage collection policies for a generational garbage collector in computer science. Using the techniques of cumulative processes, the expected costs per unit of time, i.e., expected cost rate models, are obtained, and optimal policies which minimize them are discussed analytically and computed numerically.

An initial chapter gives introduction which is constructed by review of literatures and organization of dissertation. Extended cumulative damage models in theory and their optimizations are proposed in the following chapters: Chapter 2 proposes two basic preventive maintenance policies for a used system with an initial variable damage level. Chapter 3 considers three replacement policies that are combined additive with independent damages. Chapter 4 takes up three maintenance policies for an operating system which works at random times for jobs. Chapter 5 proposes a standard cumulative damage model in which the notion of “whichever occurs last” is applied, which is called maintenance last. As applications, two stochastic models based on the working schemes of a generational garbage collector are proposed in Chapter 6. In the end of dissertation, the results are summarized and future problems are given.

The models proposed in Chapters 2–5 are derived from practical systems as introduced in every chapter and could be applied to them by suitable modifications and extensions. The theoretical methods proposed in Chapter 6 could provide some useful information to computer programmers to design more efficient collectors.

Acknowledgment

The author would like to appreciate Professor T. Nakagawa, the supervisor of my Ph.D. course, for his constant guidance, encouragement and suggestions through this dissertation.

The author wishes to thank the members of this dissertation reviewing committee: Professor T. Kondo and Professor N. Ito of Aichi Institute of Technology.

The author is grateful to Professor S. Nakamura of Kinjo Gakuin University, Professor C. Qian of Nanjing University of Technology, Professor M. Chen of Fu Jen Catholic University, and Dr. S. Mizutani of Aichi University of Technology for their encouragement and cooperation during my Ph.D. course.

The author expresses thanks to Professor T. Dohi of Hiroshima University and Professor M. Kimura of Hosei University for having presented my related research at some international conferences, and is also grateful to Professor S. Sheu of Providence University, Professor L. Cui of Beijing Institute of Technology, Professor S. Yamada of Tottori University, Professor Y. Hayakawa of Waseda University, Professor S. Chukova of Victoria University, and Professor M. J. Zuo of University of Alberta, who have helped me a lot in my research papers.

Furthermore, the author wishes to thank all members of Nagoya Computer and Reliability Research Group for their useful comments and discussions, and thank all members of Department of Business Administration, Aichi Institute of Technology, for their all kinds of help.

Finally, it is a special pleasure to thank my family for their mental and various supports.

Contents

List of Tables

List of Figures

CHAPTER 1

Introduction

1.1 Review of Literatures

Public infrastructures in advanced nations are now becoming obsolete (Hudson, et al., 1997), and the number of aged plants in Japan is increasing greatly in the near future (Nakagawa and Ito, 2008). A deliberate maintenance plan is indispensable to operate such systems without serious trouble caused by failures. That is, systems should undergo suitable maintenances at adequate times by considering both profits of operations and losses of unexpected failures or maintenances (Zhao, et al., 2012a). We call that maintenances after failure and before failure are corrective maintenance (CM) and preventive maintenance (PM), respectively (Nakagawa, 2005, p.2). CM may be costly, and sometimes requires a long time, so that how to determine the schedule of PM becomes an important problem for an operating system. However, it is not wise to maintain a system with unnecessary frequency.

A methodical survery of maintenance policies in reliability theory was done (Nakagawa, 2005). The recent published books (Osaki, 2002; Wang and Pham, 2007; Kobbacy and Murthy, 2008; Nakagawa, 2008, 2011; Manzini, et al., 2010) collected many maintenance models in theory and their applications in industrial systems. On the other hand, most systems might fail due to the damage stored within them by shocks such as jolt, stress, or environment change. This is well-known as the cumulative damage model (Cox, 1962) which plays an important role in reliability theory: The model is considered as a sequence of shocks which occur randomly in time as an event in accordance with a stochastic process and give some amount of damage to a system. The damage suffered for the system is accumulated to the current damage level and weakens the system gradually. The system fails when the total damage exceeds a failure level.

Some reliability quantities of cumulative damage models have already been obtained (Cox, 1962; Esary, et al., 1973; Nakagawa and Osaki, 1974). The first research book (Bogdanoff and Kozin, 1985) introduced some probabilistic models which are related to cumulative damage, however, the case studies for the models are few and the analyses might be too difficult theoretically to apply them to practical models. To build a bridge between theory and practice, book (Nakagawa, 2007) summarized sufficiently PM policies and their optimization problems for shock and damage models, using the techniques of stochastic processes. A variety of PM models subjected to shocks were studied extensively (Wortman, et al., 1994; Sheu, et al., 1996, 1998, 2002, 2004, 2012; Qian, et al., 2005; Zhao, et al., 2010a, 2011a, 2012a, 2012b). The damage models have been applied to garbage collection models (Satow, et al., 1996a, 1196b) by replacing shock by update and damage by garbage, backup models of database systems (Qian, et al., 1999, 2002a, 2002b, 2010; Nakamura, et al., 2003) by replacing shock by update and damage by dumped file, and software rejuvenation models (Zhao, et al., 2009) in computer sciences by replacing shock by aging-related fault and damage by consumption of physical memory.

In the computer science community, the technique of garbage collection (Jones and Lins, 1996) is one automatic process of memory recycling, which refers to that objects in the memory no longer referenced by programs are called garbage and should be thrown away. A garbage collector determines which objects are garbage and makes the heap space occupied by such garbage available again for the subsequent new objects. Garbage collection plays an important role in Java's security strategy, however, it adds a large overhead that can deteriorate the program performances. From related studies which are summarized in (Jones and Lins, 1996), a garbage collector spends between 25 and 40 percent of execution time of programs for its work in general, and delays caused by such a garbage collection are obtrusive.

With regarding to garbage collection modeling and optimization, there have been very few research papers that studied analytical expressions of optimal policies for a garbage collector. The modeling methods (Satow, et al., 1996a, 1196b) did not consider the theoretical point of garbage collection working schemes essentially. Most problems in other literatures were concerned with several ways to introduce garbage collection methods in techniques and how to tune the garbage collector by simulations, which is more complex and time consuming due to the random accesses of programs in the memory in practice (Ungar and Jackson, 1992; Kaldewaij and Vries, 2001; Lee and Chang, 2004; Clinger and Rojas, 2006; Soman and Krintz, 2007). We propose that garbage collection is a stochastic decision making process and should be analyzed by the theory of stochastic processes from the viewpoints of management. Optimal policies for a generational garbage collector with tenuring threshold and major collection times according to practical working schemes (Zhao, et al., 2010b, 2011b, 2012c) were studied recently.

1.2 Organization of Dissertation

The main body of this dissertation is divided into Introduction, Chapters 2–6, Conclusions, and Bibliography.

Chapter 2 gives a definition of a used system with an initial variable damage level Y_0, and proposes two basic imperfect PM policies which are done at a planned time T or at a shock number N. Furthermore, two extended models, by considering increasing inspection costs suffered for shocks, are formulated.

Chapter 3 proposes that the system would fail by both additive and independent damages, and considers three replacement policies with such two kinds of damages: The unit is replaced at a planned time and undergoes minimal repair when independent damage occurs. First, a standard cumulative damage model where the unit is replaced at a planned time T is considered. Second, the total damage is measured only at periodic times nT_0. Third, the total damage increases linearly with time t approximately.

Chapter 4 takes up three maintenance policies for an operating system which works at random times for jobs. First, PM is made at the Nth completion of working time, and the system fails with probability $p(x)$ when the total damage is x. Second, the system is maintained at the first completion of some working times over time T. Third, when a limit number N of working times are considered, maintenance is made at a planned time T or at a damage level Z.

Chapter 5 gives two definitions of maintenance first (MF) and maintenance last (ML), where MF has been discussed widely in literatures, and MF denotes that PM is done before failure at a planned time T, at a damage level Z, or at a shock number N, whichever occurs last. To derive the optimization problems, two alternative policies which combined time-based with condition-based PM are discussed, i.e., optimal policies of T_L^* for N, Z_L^* for T, and N_L^* for T are obtained. Comparison methods between such a ML and the conventional MF are explored.

Chapter 6 proposes two application models of cumulative damage processes to garbage collection policies in computer science, according to the practical working schemes of a generational garbage collector. We suppose that garbage collections occur at a nonhomogeneous Poisson process, and divide the collections into minor, tenuring, and major collections, respectively. Minor collections are made when the garbage collector begins to work, tenuring collection is made at a planned time T or at the first collection time when surviving objects have exceeded K, and major collection is made at time T or at the Nth collection.

In Chapters 2–6, expected cost rates for all policies are obtained, by using the techniques of cumulative processes in reliability theory. Optimal policies are discussed analytically, and numerical examples are computed when a Poisson process and exponential or normal distributions are adopted. The models proposed in Chapters 2–5 can also be applied to practical systems: Chapter 2 could be modified in garbage collection or defragmentation models in software systems when collection or defragment is imperfect. Chapter 3 could be used in reorganization models of a structural database. When the operating system is executing jobs or computer procedures successively, Chapters 4 and 5 could provide new topics and methods as practical policies.

Finally, chapter 7 summaries the results that have been obtained in this dissertation.

Chapter 2

Maintenance for a Used System

In some practical situations, it may be more economical to operate a used system than to do a new one. From this viewpoint, this chapter proposes two basic preventive maintenance policies for a used system: The system with an initial variable damage Y_0 begins to operate at time 0, and suffers damage due to shocks. It fails when the total damage exceeds a failure level K and corrective maintenance is made immediately. To prevent such a failure, it undergoes preventive maintenance at a planned time T or at a shock number N, but maintenances are imperfect. Furthermore, increasing inspection cost that is suffered for every shock is applied to the above policies in the extended models. Using the theory of cumulative processes in reliability, expected cost rate models are obtained, and optimal policies which minimize them are derived analytically and discussed numerically.

2.1 Introduction

As introduced in Chapter 1, maintenances after failure and before failure are called corrective maintenance (CM) and preventive maintenance (PM), respectively. When CM is done, it may require much more time and higher cost, so we need to do PM to prevent failure. Even so, we should not to do it too often from the viewpoints of time and cost. In this case, various PM policies and their optimizations, which make the system as good as new, including some minimal repairs, were summarized in (Nakagawa, 2005). However, CM and PM would not make a system like new but younger, i.e., maintenances are imperfect in general. Some imperfect PM models have been considered in (Chan and Downs, 1978; Murthy and Nguyen, 1981; Brown and Proschan, 1983; Wang and Pham, 2003; Nakagawa, 2005, 2007). In some practical situations, it may be more economical to operate a used system than to do a new one. Optimal replacement policies for a used system were studied in (Muth, 1977; Nakagawa, 1979; Qian, et al., 2005). However, an initial damage level of the system at time 0 or after imperfect PM may be a variable and its distribution function may be different from those of damage caused by shocks during its operation.

We suppose that a used system begins to operate at time 0, and its initial damage is a random variable Y_0 $(0 \le Y_0 \le K)$. Shocks occur at a nonhomogeneous Poisson process and each shock causes a random amount of damage to the system. These damages are accumulated to the current damage level. The system undergoes imperfect preventive maintenance (IPM) at a planned time T $(0 < T \le \infty)$, at a shock number N $(N = 1, 2, \cdots)$, or imperfect corrective maintenance (ICM) is done when the total damage exceeds a failure level K, whichever occurs first. The expected cost rates are obtained by using the techniques of cumulative damage models (Nakagawa, 2007), and optimal maintenance policies which minimize them are discussed analytically. Furthermore, increasing inspection cost that is suffered for every shock is applied to the above policies in extended models, the expected cost rates are obtained and computed numerically.

2.2 Expected Cost Rate

Suppose that shocks occur at a nonhomogeneous Poisson process with an intensity function $\lambda(t)$ and a mean-value function $R(t) \equiv \int_0^t \lambda(u)\mathrm{d}u$, i.e., $\lambda(t) \equiv R'(t)$. Then, the probability that shocks occur exactly j times in the interval $[0, t]$ is (Nakagawa, 2007, p.21)

$$H_j(t) \equiv \frac{[R(t)]^j}{j!}\mathrm{e}^{-R(t)} \qquad (j = 0, 1, 2, \cdots).$$

It is assumed that the system with an initial damage Y_0 begins to operate at time 0, where Y_0 is a random variable and has a distribution function $G_0(x) = \Pr\{Y_0 \leq x\}$ for $x \leq K$ and $G_0(x) \equiv 1$ for $x > K$ with mean $\mu_0 \equiv \int_0^K \overline{G}_0(x)\mathrm{d}x < K$, i.e., $\int_0^K G_0(x)\mathrm{d}x = K - \mu_0$. Further, an amount Y_j of damage due to the jth shock has an distribution function $G(y) \equiv \Pr\{Y_j \leq y\}$ $(j = 1, 2, \cdots)$, these damages are accumulated to the current damage level. We call the system as a used system. Then, the total damage $Z_j \equiv Y_0 + \sum_{i=1}^{j} Y_i$ $(j = 1, 2, \cdots)$ up to the jth shock, where $Z_0 \equiv Y_0$, has a distribution function

$$\Pr\{Z_j \leq w\} = \int_0^w G^{(j)}(w - x)\mathrm{d}G_0(x) \qquad (j = 0, 1, 2, \cdots), \tag{2.1}$$

where $G^{(j)}(x)$ represents the j-fold Stieltjes convolution of $G(x)$ with itself, and $G^{(0)}(x) \equiv 1$ for $x \geq 0$.

Let $Z(t)$ be the total damage at time t. Then, the distribution function of $Z(t)$ is

$$\Pr\{Z(t) \leq w\} = \sum_{j=0}^{\infty} H_j(t) \int_0^w G^{(j)}(w - x)\mathrm{d}G_0(x). \tag{2.2}$$

Suppose that the system undergoes ICM when the total damage exceeds a failure level K, and undergoes IPM at a planned time T $(0 < T \leq \infty)$ or at a shock number N $(N = 1, 2, \cdots)$, whichever occurs first. The damage level decreases to Y_0 by either IPM or ICM, i.e., the system becomes an identical system with an initial damage level Y_0 which has a general distribution $G_0(x)$. However, the cost for ICM would be higher than that for IPM, because the system might suffer serious damage when the total damage has exceeded a failure level K. Furthermore, the maintenance cost might be affected by the amount of total damage when the system undergoes ICM and IPM. From the above reasons, we introduce the following maintenance costs: Cost c_T and c_N are the respected fixed costs for IPM at time T and at shock N, and c_K is the fixed cost for ICM, where $c_T < c_K$ and $c_N < c_K$. In addition, $c_0(x)$ $(0 \leq x \leq K)$ is an additional cost when the total damage is x at maintenance time.

For the above system, the probability that the system undergoes IPM at time

T is

$$\mathrm{P}_T = \sum_{j=0}^{N-1} H_j(T) \int_0^K G^{(j)}(K-x)\mathrm{d}G_0(x), \tag{2.3}$$

and the probability that it undergoes IPM at shock N is

$$\mathrm{P}_N = \int_0^T H_{N-1}(t)\lambda(t)\mathrm{d}t \int_0^K G^{(N)}(K-x)\mathrm{d}G_0(x). \tag{2.4}$$

Thus, the expected cost when IPM is done is

$$\begin{aligned} C_{IPM} =& \sum_{j=0}^{N-1} H_j(T) \int_0^K \int_0^{K-x} [c_T + c_0(x+y)]\mathrm{d}G^{(j)}(y)\mathrm{d}G_0(x) \\ &+ \int_0^T H_{N-1}(t)\lambda(t)\mathrm{d}t \int_0^K \int_0^{K-x} [c_N + c_0(x+y)]\mathrm{d}G^{(j)}(y)\mathrm{d}G_0(x). \end{aligned} \tag{2.5}$$

The probability that the system undergoes ICM when the total damage exceeds a failure level K is

$$\mathrm{P}_K = \sum_{j=0}^{N-1} \int_0^T H_j(t)\lambda(t)\mathrm{d}t \int_0^K \int_0^{K-x} \overline{G}(K-x-y)\mathrm{d}G^{(j)}(y)\mathrm{d}G_0(x), \tag{2.6}$$

where $\overline{\Phi}(x) \equiv 1 - \Phi(x)$ for any function $\Phi(x)$, and the probability that the system undergoes ICM when the total damage exceeds K at shock N is included in (2.6) because it has become the failure state. Note that $\mathrm{P}_T + \mathrm{P}_N + \mathrm{P}_K \equiv 1$. Thus, the expected cost when ICM is done is

$$C_{ICM} = [c_K + c_0(K)] \sum_{j=0}^{N-1} \int_0^T H_j(t)\lambda(t)\mathrm{d}t \int_0^K \int_0^{K-x} \overline{G}(K-x-y)\mathrm{d}G^{(j)}(y)\mathrm{d}G_0(x). \tag{2.7}$$

The mean time to maintenance is

$$\begin{aligned} E(L) =& T\sum_{j=0}^{N-1} H_j(T) \int_0^K G^{(j)}(K-x)\mathrm{d}G_0(x) \\ &+ \int_0^T tH_{N-1}(t)\lambda(t)\mathrm{d}t \int_0^K G^{(N)}(K-x)\mathrm{d}G_0(x) \\ &+ \sum_{j=0}^{N-1} \int_0^T tH_j(t)\lambda(t)\mathrm{d}t \int_0^K \int_0^{K-x} \overline{G}(K-x-y)\mathrm{d}G^{(j)}(y)\mathrm{d}G_0(x) \end{aligned}$$

$$= \sum_{j=0}^{N-1} \int_0^T H_j(t)\mathrm{d}t \int_0^K G^{(j)}(K-x)\mathrm{d}G_0(x). \tag{2.8}$$

Therefore, the expected cost rate is, from (2.5), (2.7), and (2.8),

$$C(T,N) = \frac{\begin{array}{l} c_K + c_0(K) - (c_K - c_T)\sum_{j=0}^{N-1} H_j(T)\int_0^K G^{(j)}(K-x)\mathrm{d}G_0(x) \\ -(c_K - c_N)\int_0^T H_{N-1}(t)\lambda(t)\mathrm{d}t\int_0^K G^{(N)}(K-x)\mathrm{d}G_0(x) \\ -\sum_{j=0}^{N-1} H_j(T)\int_0^K\int_x^K G^{(j)}(u-x)\mathrm{d}c_0(u)\mathrm{d}G_0(x) \\ -\int_0^T H_{N-1}(t)\lambda(t)\mathrm{d}t\int_0^K\int_x^K G^{(N)}(u-x)\mathrm{d}c_0(u)\mathrm{d}G_0(x) \end{array}}{\sum_{j=0}^{N-1}\int_0^T H_j(t)\mathrm{d}t\int_0^K G^{(j)}(K-x)\mathrm{d}G_0(x)}. \tag{2.9}$$

2.3 Optimal Policies

2.3.1 Planned Time

Suppose that the system undergoes IPM only at time T ($0 < T \leq \infty$) and ICM when the total damage exceeds a failure level K, whichever occurs first. Then, putting that $N = \infty$ in (2.9), the expected cost rate is

$$C(T) = \frac{\begin{array}{l} c_K + c_0(K) - (c_K - c_T)\sum_{j=0}^{\infty} H_j(T)\int_0^K G^{(j)}(K-x)\mathrm{d}G_0(x) \\ -\sum_{j=0}^{\infty} H_j(T)\int_0^K\int_x^K G^{(j)}(u-x)\mathrm{d}c_0(u)\mathrm{d}G_0(x) \end{array}}{\sum_{j=0}^{\infty}\int_0^T H_j(t)\mathrm{d}t\int_0^K G^{(j)}(K-x)\mathrm{d}G_0(x)}. \tag{2.10}$$

We seek an optimal time T^* that minimizes $C(T)$ in (2.10). Differentiating $C(T)$ with respect to T and setting it equal to zero,

$$\begin{aligned} &\lambda(T)\{(c_K - c_T)[1 - Q(T)] + P(T)\}\sum_{j=0}^{\infty}\int_0^T H_j(t)\mathrm{d}t\int_0^K G^{(j)}(K-x)\mathrm{d}G_0(x) \\ &+ (c_K - c_T)\sum_{j=0}^{\infty} H_j(T)\int_0^K G^{(j)}(K-x)\mathrm{d}G_0(x) \\ &+ \sum_{j=0}^{\infty} H_j(T)\int_0^K\int_x^K G^{(j)}(u-x)\mathrm{d}c_0(u)\mathrm{d}G_0(x) = c_K + c_0(K), \end{aligned} \tag{2.11}$$

where

$$P(T) \equiv \frac{\sum_{j=0}^{\infty} H_j(T)\int_0^K\int_x^K [G^{(j)}(u-x) - G^{(j+1)}(u-x)]\mathrm{d}c_0(u)\mathrm{d}G_0(x)}{\sum_{j=0}^{\infty} H_j(T)\int_0^K G^{(j)}(K-x)\mathrm{d}G_0(x)},$$

$$Q(T) \equiv \frac{\sum_{j=0}^{\infty} H_j(T) \int_0^K G^{(j+1)}(K-x)\mathrm{d}G_0(x)}{\sum_{j=0}^{\infty} H_j(T) \int_0^K G^{(j)}(K-x)\mathrm{d}G_0(x)}.$$

If there exists T^* which minimizes $C(T)$, it must be satisfied (2.11).

It is assumed that shocks occur in a Poisson process with rate λ, the amount of damage due to each shock has an exponential distribution with mean μ, and $c_0(x)$ is proportional to the total damage x, i.e., $H_j(t) = [(\lambda t)^j/j!]\mathrm{e}^{-\lambda t}$, $G^{(j)}(x) = \sum_{i=j}^{\infty}[(x/\mu)^i/i!]\mathrm{e}^{-x/\mu}$, and $c_0(x) = c_0 x$. Then, $P(T) = c_0\mu Q(T)$, and (2.11) becomes

$$\begin{aligned}
&\lambda[(c_K - c_T) - (c_K - c_T - c_0\mu)Q_1(T)] \sum_{j=0}^{\infty} \int_0^T H_j(t)\mathrm{d}t \int_0^K G^{(j)}(K-x)\mathrm{d}G_0(x) \\
&+ (c_K - c_T) \sum_{j=0}^{\infty} H_j(T) \int_0^K G^{(j)}(K-x)\mathrm{d}G_0(x) \\
&+ c_0 \sum_{j=0}^{\infty} H_j(T) \int_0^K \int_x^K G^{(j)}(u-x)\mathrm{d}u\mathrm{d}G_0(x) = c_K + c_0 K. \qquad (2.12)
\end{aligned}$$

Denote the left-hand side in (2.12) be $U(T)$, because $\int_0^K G_0(x)\mathrm{d}x = K - \mu_0$,

$$\lim_{T\to 0} U_1(T) = c_K - c_T + c_0(K - \mu_0) < c_K + c_0 K,$$

and $M(x) \equiv \sum_{j=1}^{\infty} G^{(j)}(x) = x/\mu$ and $\lim_{T\to\infty} Q(T) = 1$,

$$\begin{aligned}
\lim_{T\to\infty} U(T) &= (c_K - c_T) \int_0^K [1 + M(K-x)]\mathrm{d}G_0(x) \\
&= (c_K - c_T)\left[1 + \frac{1}{\mu}\int_0^K G_0(x)\mathrm{d}x\right] = (c_K - c_T)\left(1 + \frac{K-\mu_0}{\mu}\right).
\end{aligned}$$

Differentiating $U(T)$ with respect to T,

$$\frac{U'(T)}{\lambda} = -(c_K - c_T - c_0\mu)Q'(T) \sum_{j=0}^{\infty} \int_0^T H_j(t)\mathrm{d}t \int_0^K G^{(j)}(K-x)\mathrm{d}G_0(x).$$

Thus, if $Q'(T) < 0$ and $c_K - c_T - c_0\mu > 0$, $U(T)$ is a strictly increasing function of T, and hence, if a solution T^* to (2.12) exists, it is unique. It is clear that the necessity of optimal maintenance policy is that cost c_K for ICM should be greater than $c_T + c_0\mu$ for IPM, which represents the total cost of a fixed cost for IPM and the maintenance cost for a mean initial damage. Note that $1 - Q(T)$ means physically the probability of failure at the $(j+1)$th $(j=0,1,2,\cdots)$ shock in time T, given that the system has not failed at the jth shock. Thus, the condition that a finite T^* satisfies (2.12) is that $1-Q(T)$ increases strictly, i.e., $Q'(T)<0$.

In particular, when $Y_0=z_0$ $(0<z_0<K)$, i.e., $G_0(x)\equiv 1$ for $x\geq z_0$, 0 for $x<z_0$, it is proved that $Q'(T)<0$. Thus, if $c_K-c_T>\mu(c_T+c_0K)/(K-z_0)$, then there exists a finite and unique T^* $(0<T^*<\infty)$ which satisfies (2.12), and the resulting cost rate is

$$\frac{C(T^*)}{\lambda}=(c_K-c_T)-(c_K-c_T-c_0\mu)Q(T^*). \tag{2.13}$$

Next, suppose that $G_0(x)=(1-\mathrm{e}^{-x/z_0})/(1-\mathrm{e}^{-K/z_0})$ for $x\leq K$, 1 for $x>K$, i.e., $\mu_0=z_0-K\mathrm{e}^{-K/z_0}/(1-\mathrm{e}^{-K/z_0})$. It can be proved from Appendix that $Q(T)$ decreases strictly with T, and so that, $U(T)$ increases strictly with T. Therefore, we have the following optimal policy:

1. If $c_K-c_T>\mu(c_T+c_0K)/(K-\mu_0)$, then there exists a finite and unique T^* $(0<T^*<\infty)$ which satisfies (2.12), and the resulting cost rate is given in (2.13).

2. If $c_K-c_T\leq\mu(c_T+c_0K)/(K-\mu_0)$, then $T^*=\infty$, and

$$\frac{C(\infty)}{\lambda}=\frac{c_K+c_0K}{1+(K-\mu_0)/\mu}. \tag{2.14}$$

2.3.2 Shock Number

Suppose that the system undergoes IPM only at shock N $(N=1,2,\cdots)$ and ICM when the total damage exceeds a failure level K, whichever occurs first. Then, putting that $T=\infty$ in (2.9), the expected cost rate is

$$C(N)=\frac{\begin{array}{c}c_K+c_0(K)-(c_K-c_N)\int_0^K G^{(N)}(K-x)\mathrm{d}G_0(x)\\ -\int_0^K\int_x^K G^{(N)}(u-x)\mathrm{d}c_0(u)\mathrm{d}G_0(x)\end{array}}{\sum_{j=0}^{N-1}\int_0^\infty H_j(t)\mathrm{d}t\int_0^K G^{(j)}(K-x)\mathrm{d}G_0(x)}\qquad(N=1,2,\cdots). \tag{2.15}$$

We seek an optimal number N^* that minimizes $C(N)$ in (2.15). From the inequality $C(N+1) - C(N) \geq 0$,

$$\{(c_K - c_N)[1 - Q(N)] + P(N)\} \frac{\sum_{j=0}^{N-1} \int_0^\infty H_j(t)\mathrm{d}t \int_0^K G^{(j)}(K-x)\mathrm{d}G_0(x)}{\int_0^\infty H_N(t)\mathrm{d}t}$$
$$+ (c_K - c_N) \int_0^K G^{(N)}(K-x)\mathrm{d}G_0(x)$$
$$+ \int_0^K \int_x^K G^{(N)}(u-x)\mathrm{d}c_0(u)\mathrm{d}G_0(x) \geq c_K + c_0(K), \tag{2.16}$$

where

$$P(N) \equiv \frac{\int_0^K \int_x^K [G^{(N)}(u-x) - G^{(N+1)}(u-x)]\mathrm{d}c_0(u)\mathrm{d}G_0(x)}{\int_0^K G^{(N)}(K-x)\mathrm{d}G_0(x)},$$
$$Q(N) \equiv \frac{\int_0^K G^{(N+1)}(K-x)\mathrm{d}G_0(x)}{\int_0^K G^{(N)}(K-x)\mathrm{d}G_0(x)}.$$

If there exists N^* which minimizes $C(N)$, it must be satisfied (2.16). It is clear that the necessity of optimal maintenance policy is that cost c_K for ICM should be greater than $c_N + c_0\mu$ for IPM. Note that $1-Q(N)$ means physically the probability of failure at the $(N+1)$th shock, given that the system has not failed at the Nth shock. Thus, the condition that a finite N^* satisfies (2.16) is that $1-Q(N)$ increases strictly.

The failure rate plays an important role of deriving analytically optimal policies for maintenance models (Nakagawa, 2005). The functions $1 - Q(T)$ in (2.11) and $1 - Q(N)$ in (2.16) correspond to the failure rates with continuous and discrete times, respectively, and would increase strictly when finite T^* and N^* exist.

We make the similar assumptions in Section 2.3.1, then $P(N) = c_0\mu Q(N)$, and (2.16) becomes

$$[(c_K - c_N) - (c_K - c_N - c_0\mu)Q_2(N)] \sum_{j=0}^{N-1} \int_0^K G^{(j)}(K-x)\mathrm{d}G_0(x)$$
$$+ (c_K - c_N) \int_0^K G^{(N)}(K-x)\mathrm{d}G_0(x) + c_0 \int_0^K \int_x^K G^{(N)}(u-x)\mathrm{d}u\mathrm{d}G_0(x)$$
$$\geq c_K + c_0 K. \tag{2.17}$$

Denote the left-hand side in (2.17) be $U(N)$,

$$\begin{aligned}\lim_{N\to\infty} U(N) =&(c_K - c_N)\int_0^K [1+M(K-x)]\mathrm{d}G_0(x)\\ =&(c_K - c_N)\left(1+\frac{K-\mu_0}{\mu}\right),\\ U(N+1)-U(N) =& -(c_K - c_N - c_0\mu)[Q(N+1)-Q(N)]\\ &\times\sum_{j=0}^{N-1}\int_0^K G^{(j)}(K-x)\mathrm{d}G_0(x).\end{aligned}$$

Thus, if $Q(N)$ is a decreasing function of N and $c_K - c_N - c_0\mu > 0$, $U(N)$ is a increasing function of N, and hence, if a solution N^* to (2.17) exists, its minimum is unique.

In particular, when $Y_0 = z_0$ $(0 < z_0 < K)$, i.e., $G_0(x) \equiv 1$ for $x \geq z_0$, 0 for $x < z_0$, it is proved that $Q(N)$ is a decreasing function of N. Thus, if $c_K - c_N > \mu(c_N + c_0K)/(K - z_0)$, there exists a finite and unique minimum N^* $(1 \leq N^* < \infty)$ which satisfies (2.17), and the resulting cost rate is

$$(c_K - c_N - c_0\mu)Q(N^*) \leq (c_K - c_N) - \frac{C(N^*)}{\lambda} < (c_K - c_N - c_0\mu)Q(N^*-1). \tag{2.18}$$

Next, suppose that $G_0(x) = (1 - \mathrm{e}^{-x/z_0})/(1 - \mathrm{e}^{-K/z_0})$ for $x \leq K$, 1 for $x > K$. It can be proved from Appendix that $Q(N)$ decreases strictly with N, and so that, $U(N)$ increases strictly with N. Therefore, we have the following optimal policy:

1. If $c_K - c_N > \mu(c_N + c_0K)/(K - \mu_0)$, then there exists a finite and unique minimum N^* $(1 \leq N^* < \infty)$ which satisfies (2.17), and the resulting cost rate is given in (2.18).

2. If $c_K - c_N \leq \mu(c_N + c_0K)/(K - \mu_0)$, then $N^* = \infty$, and the resulting cost rate is given in (2.14).

2.3.3 Numerical Examples

Suppose that $G(x) = 1 - \mathrm{e}^{-x/\mu}$ and $G_0(x) = (1 - \mathrm{e}^{-x/z_0})/(1 - \mathrm{e}^{-K/z_0})$ for $x \leq K$. We compute the optimal policies numerically when $\mu = 1$, $c_0 = 0$ and $K = 20$.

Table 2.1 presents optimal λT^* and $C(T^*)/(\lambda c_T)$ for $c_K/c_T = 5$, 10, 20, 50 and $z_0 = 1$, 5, 10. This indicates that T^* decreases when c_K/c_T or z_0 increases, $C(T^*)$ increases when c_K/c_T or z_0 increases. Table 2.2 presents optimal N^* and $C(N^*)/(\lambda c_N)$ for $c_K/c_N = 5$, 10, 20, 50 and $z_0 = 1$, 5, 10. This shows the similar tendencies to Table 2.1 for N^* and $C(N^*)$. It is of interest that the order of the expected cost rates is $C(T^*) > C(N^*)$ for the same value of z_0 and $c_T = c_N$.

Table 2.1: Optimal λT^* and $C(T^*)/(\lambda c_T)$ for c_K/c_T and z_0.

c_K/c_T	$z_0 = 1$		$z_0 = 5$		$z_0 = 10$	
	λT^*	$C(T^*)/(\lambda c_T)$	λT^*	$C(T^*)/(\lambda c_T)$	λT^*	$C(T^*)/(\lambda c_T)$
5	11.2	0.1139	8.8	0.1498	7.6	0.1839
10	9.4	0.1336	7.3	0.1805	6.1	0.2267
20	8.2	0.1542	6.1	0.2135	4.9	0.2739
50	6.7	0.1840	4.9	0.2638	4.0	0.3481

Table 2.2: Optimal N^* and $C(N^*)/(\lambda c_N)$ for c_K/c_N and z_0.

c_K/c_N	$z_0 = 1$		$z_0 = 5$		$z_0 = 10$	
	N^*	$C(N^*)/(\lambda c_N)$	N^*	$C(N^*)/(\lambda c_N)$	N^*	$C(N^*)/(\lambda c_N)$
5	12	0.0952	10	0.1241	8	0.1510
10	11	0.1060	8	0.1405	7	0.1741
20	10	0.1169	7	0.1593	6	0.1981
50	9	0.1322	6	0.1843	5	0.2334

We could explain the optimal policies as follows: (1) When cost c_K for ICM increases, we should advance the time of IPM, that is, T^* and N^* should be decreased, in order to reduce the probability of failure. (2) When z_0 increases, i.e., a used system begins to operate at time 0 with a higher damage, then, its life will be shorter due to shocks, and so that, T^* and N^* should be advanced. (3) Compare the numerical examples above, concrete performances of two policies would be depend on maintenance costs, system structures and environment, maintenance engineers, and so on. Take into such considerations, we would adopt which policy is suitable for an actual system. That is, from the viewpoint of economy, the policy N^* is better than that of T^*. However, from the viewpoint of simplicity of operation, the policy T^* would be better because we do not need to count the number of shocks.

2.4 Extended Models

2.4.1 Expected Cost Rates

Introduce the cost c_{Mi} suffered for the ith shock, where $0 < c_{M1} \leq c_{M2} \leq \cdots \leq c_{Mi} \leq \cdots$. For example, this would be the inspection cost of measuring total damage level or the cost of some treatment for each shock, and would be usually much smaller compared to PM costs c_T and c_N.

First, consider that the system undergoes IPM at time T $(0 < T \leq \infty)$ and ICM when the total damage exceeds a failure level K, whichever occurs first. Then, the total expected cost for each shock before any maintenance is

$$\begin{aligned} C_M =& \sum_{j=1}^{\infty}\sum_{i=1}^{j} c_{Mi} H_j(T) \int_0^K G^{(j)}(K-x)\mathrm{d}G_0(x) \\ &+ \sum_{j=1}^{\infty}\sum_{i=1}^{j} c_{Mi} \int_0^T H_j(t)\lambda(t)\mathrm{d}t \int_0^K \int_0^{K-x} \overline{G}(K-x-y)\mathrm{d}G^{(j)}(y)\mathrm{d}G_0(x) \\ =& \sum_{j=0}^{\infty} c_{Mj+1} \int_0^T H_j(t)\lambda(t)\mathrm{d}t \int_0^K G^{(j+1)}(K-x)\mathrm{d}G_0(x), \end{aligned} \tag{2.19}$$

From (2.10) and (2.19), the expected cost rate is

$$\widehat{C}(T) = \frac{\begin{array}{l} c_K + c_0(K) - (c_K - c_T)\sum_{j=0}^{\infty} H_j(T)\int_0^K G^{(j)}(K-x)\mathrm{d}G_0(x) \\ -\sum_{j=0}^{\infty} H_j(T)\int_0^K\int_x^K G^{(j)}(u-x)\mathrm{d}c_0(u)\mathrm{d}G_0(x) \\ +\sum_{j=0}^{\infty} c_{Mj+1}\int_0^T H_j(t)\lambda(t)\mathrm{d}t\int_0^K G^{(j+1)}(K-x)\mathrm{d}G_0(x) \end{array}}{\sum_{j=0}^{\infty}\int_0^T H_j(t)\mathrm{d}t\int_0^K G^{(j)}(K-x)\mathrm{d}G_0(x)}. \tag{2.20}$$

Second, consider that the system undergoes IPM at shock N $(N = 1, 2, \cdots)$ and ICM when the total damage exceeds a failure level K, whichever occurs first.

Then, the total expected cost for each shock before any maintenance is

$$\begin{aligned} C_M = & \sum_{i=1}^{N-1} c_{Mi} \int_0^K G^{(N)}(K-x)\mathrm{d}G_0(x) \\ & + \sum_{j=1}^{N-1}\sum_{i=1}^{j} c_{Mi} \int_0^K \int_0^{K-x} \overline{G}(K-x-y)\mathrm{d}G^{(j)}(y)\mathrm{d}G_0(x) \\ = & \sum_{j=1}^{N-1} c_{Mj} \int_0^K G^{(j)}(K-x)\mathrm{d}G_0(x), \end{aligned} \tag{2.21}$$

where $\sum_{j=1}^{0} = 0$. From (2.15) and (2.21), the expected cost rate is

$$\widehat{C}(N) = \frac{\begin{array}{l} c_K + c_0(K) - (c_K - c_N)\int_0^K G^{(N)}(K-x)\mathrm{d}G_0(x) \\ -\int_0^K \int_x^K G^{(N)}(u-x)\mathrm{d}c_0(u)\mathrm{d}G_0(x) \\ +\sum_{j=1}^{N-1} c_{Mj}\int_0^K G^{(j)}(K-x)\mathrm{d}G_0(x) \end{array}}{\sum_{j=0}^{N-1}\int_0^\infty H_j(t)\mathrm{d}t \int_0^K G^{(j)}(K-x)\mathrm{d}G_0(x)} \quad (N = 1, 2, \cdots). \tag{2.22}$$

2.4.2 Numerical Examples

Suppose that $c_0(x) = 0$, $c_{Mj} = jc_M$, $H_j(t) = [(\lambda t)^j/j!]\mathrm{e}^{-\lambda t}$, $G(x) = 1 - \mathrm{e}^{-x/\mu}$ and $G_0(x) = (1 - \mathrm{e}^{-x/z_0})/(1 - \mathrm{e}^{-K/z_0})$. Then, we compute optimal $\widehat{T}^*$ and $\widehat{N}^*$, and $\widehat{C}(\widehat{T}^*)/c_T$ and $\widehat{C}(\widehat{N}^*)/c_N$ numerically when $K = 20$, $z_0 = 1$, $\lambda = 1$, $\mu = 1$.

Table 2.3 presents optimal $\widehat{T}^*$ and $\widehat{C}(\widehat{T}^*)/c_T$ for $c_M/c_T = 0.01$, 0.02, 0.1 and $c_K/c_T = 5$, 10, 20, 50. This indicates that $\widehat{T}^*$ decreases when c_K/c_T or c_M/c_T increases, $\widehat{C}(\widehat{T}^*)$ increases when c_K/c_T or c_M/c_T increases. Table 2.4 presents optimal $\widehat{N}^*$ and $\widehat{C}(\widehat{N}^*)/c_N$ for $c_M/c_N = 0.01$, 0.02, 0.1 and $c_K/c_N = 5$, 10, 20, 50. This shows the similar tendencies to Table 2.3 for $\widehat{N}^*$ and $\widehat{C}(\widehat{N}^*)$, but if c_M/c_N is very larger, when c_K/c_N increases, $\widehat{N}^*$ and $\widehat{C}(\widehat{N}^*)/c_N$ will be stable. It is of interest that the order of the expected cost rates is $\widehat{C}(\widehat{T}^*) > \widehat{C}(\widehat{N}^*)$ for the same parameters.

It could be explained as follows: (1) Compared with those in Section 2.3.3, optimal maintenance times are advanced due to shocks. (2) When cost c_K for ICM increases, we should advance the time of IPM, the reason is the same as that in Section 2.3.3. (3) When c_M/c_T or c_M/c_N increases, it means unit cost for shocks will increase, so that IPM should be advanced to reduce the total expected cost for shocks. (4) When the inspection cost for shocks is large, cost for ICM will have no effect on the optimal policies. It is interest of that for the two policies, both optimal policies and resulting cost rates are stable at the similar level when the inspection cost for shocks is large.

Table 2.3: Optimal $\widehat{T}^*$ and $\widehat{C}(\widehat{T}^*)/c_T$ for c_M/c_T and c_K/c_T.

c_K/c_T	$c_M/c_T = 0.01$		$c_M/c_T = 0.02$		$c_M/c_T = 0.1$	
	$\widehat{T}^*$	$\widehat{C}(\widehat{T}^*)/c_T$	$\widehat{T}^*$	$\widehat{C}(\widehat{T}^*)/c_T$	$\widehat{T}^*$	$\widehat{C}(\widehat{T}^*)/c_T$
5	9.7	0.1752	8.5	0.2309	4.6	0.5527
10	8.5	0.1882	7.9	0.2391	4.6	0.5533
20	7.6	0.2032	7.0	0.2499	4.3	0.5545
50	6.4	0.2271	6.1	0.2688	4.3	0.5574

Table 2.4: Optimal $\widehat{N}^*$ and $\widehat{C}(\widehat{N}^*)/c_N$ for c_M/c_N and c_K/c_N.

c_K/c_N	$c_M/c_N = 0.01$		$c_M/c_N = 0.02$		$c_M/c_N = 0.1$	
	$\widehat{N}^*$	$\widehat{C}(\widehat{N}^*)/c_N$	$\widehat{N}^*$	$\widehat{C}(\widehat{N}^*)/c_N$	$\widehat{N}^*$	$\widehat{C}(\widehat{N}^*)/c_N$
5	11	0.1476	9	0.1928	4	0.4000
10	10	0.1530	9	0.1950	4	0.4000
20	9	0.1593	8	0.1986	4	0.4000
50	8	0.1693	8	0.1993	4	0.4000

2.5 Concluding Remarks

We have discussed two imperfect preventive maintenance policies for a used system at a planned time T and at a shock number N for basic models and introduced extra inspection cost for each shock as one of extended models. Expected cost rates are obtained by using the techniques of cumulative processes in reliability theory. Optimal policies of T^* and N^* which minimize them are derived analytically for basic models, and optimal $\widehat{T}^*$ and $\widehat{N}^*$ are computed numerically for the extended models. Useful discussions for such results are given.

From analytical discussions in optimizations, we have found that $1-Q(T)$ and $1-Q(N)$ which have the physical meanings of failure rates with continuous and discrete times play an important role in deriving optimal policies, and the necessity of optimizations is also that cost for ICM should be greater than that for the first IPM which includes the maintenance cost for the initial damage. From numerical analyses, it has been shown that how the initial damage level z_0 and the inspection cost c_M affect the optimal times. By comparing numerical T^* with N^* or $\widehat{T}^*$ with $\widehat{N}^*$, if we adopt the policy from the viewpoint of economy, the policy N^* is better than that of T^*, but if from the viewpoint of simplicity of operation, the policy T^* would be better because we do not need to count the number of shocks.

As introduced in Chapter 1, the damage models have been applied to garbage collection models, backup models, and software rejuvenation models in computer sciences. The method proposed in this chapter could be applied to garbage collection or defragmentation models in software systems. As high information has been developed in the modern society, software always has to work for $24 * 7$ hours with non-stop service, application programs could not collect garbage or defragment in software systems perfectly in time. The models with initial damage proposed in this chapter could be applied to such models, by modifying and extending them suitably.

Appendix

When $G^{(j)}(x) = \sum_{i=j}^{\infty}[(x/\mu)^i/i!]\mathrm{e}^{-x/\mu}$ $(j = 0, 1, 2, \cdots)$ and $G_0(x) = (1-\mathrm{e}^{-x/z_0})/(1-\mathrm{e}^{-K/z_0})$ for $x \leq K$, prove that: 1. $1 - Q(j)$ increases strictly with j; 2. $Q(T)$ decreases strictly with T, where

$$1 - Q(j) = \frac{\int_0^K [G^{(j)}(K-x) - G^{(j+1)}(K-x)]\mathrm{d}G_0(x)}{\int_0^K G^{(j)}(K-x)\mathrm{d}G_0(x)}, \tag{A.1}$$

$$Q(T) = \frac{\sum_{j=0}^{\infty} \frac{(\lambda T)^j}{j!} \int_0^K G^{(j+1)}(K-x)\mathrm{d}G_0(x)}{\sum_{j=0}^{\infty} \frac{(\lambda T)^j}{j!} \int_0^K G^{(j)}(K-x)\mathrm{d}G_0(x)}. \tag{A.2}$$

1. Prove that

$$\frac{\int_0^K \frac{(x/\mu)^j}{j!}\mathrm{e}^{-x/\mu}\mathrm{e}^{x/z_0}\mathrm{d}x}{\sum_{i=j}^{\infty}\int_0^K \frac{(x/\mu)^i}{i!}\mathrm{e}^{-x/\mu}\mathrm{e}^{x/z_0}\mathrm{d}x} < \frac{\int_0^K \frac{(x/\mu)^{j+1}}{(j+1)!}\mathrm{e}^{-x/\mu}\mathrm{e}^{x/z_0}\mathrm{d}x}{\sum_{i=j+1}^{\infty}\int_0^K \frac{(x/\mu)^i}{i!}\mathrm{e}^{-x/\mu}\mathrm{e}^{x/z_0}\mathrm{d}x}.$$

Denote

$$\begin{aligned}\Phi(K) \equiv & \int_0^K \frac{(x/\mu)^{j+1}}{(j+1)!}\mathrm{e}^{-x/\mu}\mathrm{e}^{x/z_0}\mathrm{d}x \sum_{i=j}^{\infty}\int_0^K \frac{(x/\mu)^i}{i!}\mathrm{e}^{-x/\mu}\mathrm{e}^{x/z_0}\mathrm{d}x \\ & - \int_0^K \frac{(x/\mu)^j}{j!}\mathrm{e}^{-x/\mu}\mathrm{e}^{x/z_0}\mathrm{d}x \sum_{i=j+1}^{\infty}\int_0^K \frac{(x/\mu)^i}{i!}\mathrm{e}^{-x/\mu}\mathrm{e}^{x/z_0}\mathrm{d}x, \end{aligned} \tag{2.23}$$

where $\Phi(0) = 0$. Differentiating $\Phi(K)$ with respect to K,

$$\begin{aligned}\Phi'(K) = & \sum_{i=j}^{\infty} \frac{\mathrm{e}^{-K/(\mu-z_0)}}{(j+1)!(i+1)!}\int_0^K \mathrm{e}^{-x/(\mu-z_0)}\Big\{(i+1)\Big[(K/\mu)^{j+1}(x/\mu)^i \\ & + (x/\mu)^{j+1}(K/\mu)^i\Big] - (j+1)\Big[(K/\mu)^j(x/\mu)^{i+1} + (x/\mu)^j(K/\mu)^{i+1}\Big]\Big\}\mathrm{d}x \\ \geq & \sum_{i=j}^{\infty}\frac{\mu\mathrm{e}^{-2K/(\mu-z_0)}}{(j+1)!(i+1)!}\left\{(i+1)\left[\frac{(K/\mu)^{j+1}(K/\mu)^{i+1}}{i+1} + \frac{(K/\mu)^i(K/\mu)^{j+2}}{j+2}\right]\right. \\ & \left. -(j+1)\left[\frac{(K/\mu)^j(K/\mu)^{i+2}}{i+2} + \frac{(K/\mu)^{i+1}(K/\mu)^{j+1}}{j+1}\right]\right\} \\ = & \sum_{i=j}^{\infty}\frac{\mu\mathrm{e}^{-2K/(\mu-z_0)}K^{i+j+2}}{(j+1)!(i+1)!}\left(\frac{i+1}{j+2} - \frac{j+1}{i+2}\right) > 0, \end{aligned}$$

which completes that $Q(j)$ decreases strictly with j.

2. Denote $F_j(K) \equiv \int_0^K G^{(j)}(K-x)\mathrm{d}G_0(x)$. Differentiating $Q(T)$ with respect to T,

$$\begin{aligned} Q'(T) = & \frac{\lambda}{\left\{\sum_{j=0}^{\infty}[(\lambda T)^j/j!]F_j(K)\right\}^2}\left[\sum_{j=0}^{\infty}\frac{(\lambda T)^j}{j!}F_{j+2}(K)\sum_{j=0}^{\infty}\frac{(\lambda T)^j}{j!}F_j(K)\right. \\ & \left. - \sum_{j=0}^{\infty}\frac{(\lambda T)^j}{j!}F_{j+1}(K)\sum_{j=0}^{\infty}\frac{(\lambda T)^j}{j!}F_{j+1}(K)\right] \\ = & \frac{\lambda}{\left\{\sum_{j=0}^{\infty}[(\lambda T)^j/j!]F_j(K)\right\}^2}\left[\sum_{j=0}^{\infty}j\frac{(\lambda T)^j}{j!}F_{j+1}(K)\sum_{j=-1}^{\infty}\frac{(\lambda T)^j}{(j+1)!}F_{j+1}(K)\right. \end{aligned}$$

$$- \sum_{j=0}^{\infty} \frac{(\lambda T)^j}{j!} F_{j+1}(K) \sum_{j=0}^{\infty} \frac{(\lambda T)^j}{j!} F_{j+1}(K) \Bigg] .$$

Thus, the bracket on the right-hand side is

$$\begin{aligned}
&\sum_{j=0}^{\infty} j \frac{(\lambda T)^j}{j!} F_{j+1}(K) \sum_{j=-1}^{\infty} \frac{(\lambda T)^j}{(j+1)!} F_{j+1}(K) - \sum_{j=0}^{\infty} \frac{(\lambda T)^j}{j!} F_{j+1}(K) \sum_{j=0}^{\infty} \frac{(\lambda T)^j}{j!} F_{j+1}(K) \\
&= \sum_{i=0}^{\infty} i \frac{(\lambda T)^i}{i!} F_{i+1}(K) \sum_{j=-1}^{\infty} \frac{(\lambda T)^j}{(j+1)!} F_{j+1}(K) \\
&\quad - \sum_{i=0}^{\infty} \frac{(\lambda T)^i}{i!} F_{i+1}(K) \sum_{j=-1}^{\infty} (j+1) \frac{(\lambda T)^j}{(j+1)!} F_{j+1}(K) \\
&= \sum_{j=-1}^{\infty} \frac{(\lambda T)^j}{(j+1)!} F_{j+1}(K) \left\{ \sum_{i=0}^{j+1} \frac{(\lambda T)^i}{i!} F_{i+1}(K) [i - (j+1)] \right. \\
&\quad \left. + \sum_{i=j+1}^{\infty} \frac{(\lambda T)^i}{i!} F_{i+1}(K) [i - (j+1)] \right\} \\
&= \sum_{j=0}^{\infty} \frac{(\lambda T)^{j-1}}{j!} F_j(K) \sum_{i=0}^{j} \frac{(\lambda T)^i}{i!} F_{i+1}(K)(i-j) \\
&\quad + \sum_{i=0}^{\infty} \frac{(\lambda T)^i}{i!} F_{i+1}(K) \sum_{j=0}^{i} \frac{(\lambda T)^{j-1}}{j!} F_j(K)(i-j) \\
&= \sum_{j=0}^{\infty} \frac{(\lambda T)^j}{j!} \sum_{i=0}^{j} \frac{(\lambda T)^{i-1}}{i!} F_i(K) F_j(K)(i-j) \left(\frac{F_{i+1}(K)}{F_i(K)} - \frac{F_{j+1}(K)}{F_j(K)} \right) < 0,
\end{aligned}$$

which proves 2, because $F_{j+1}(K)/F_j(K)$ decreases strictly with j from proof 1.

Chapter 3

Additive and Independent Damages

Most systems would fail roughly with time by both causes of additive and independent damages. From such a viewpoint, this chapter considers three combined replacement policies with two kinds of damages: The unit is replaced at a planned time or when the total additive damage exceeds a failure level, whichever occurs first, and undergoes minimal repair when independent damage occurs. First, a standard cumulative damage model where the unit suffers some damage due to shocks and the total damage is additive is considered. Second, the total damage is measured only at periodic times. Third, the total damage increases linearly with time t approximately. Using the theory of cumulative processes, expected cost rates are obtained, and optimal policies which minimize them are derived analytically. Finally, optimal policies are computed and compared numerically, and useful discussions for such results are given.

3.1 Introduction

Most systems might fail due to the total damage stored within them by shocks such as jolt, stress, or environment change, which is called cumulative damage process, as introduced in Chapter 1. On the other hand, when the total damage is not additive, the unit fails when the damage due to some shock has exceeded a failure level. This is called the independent damage model, and its typical examples are the fracture of brittle materials such as glasses (Esary, et al., 1973), and semiconductor parts which have failed by some overcurrent or fault voltage. So that, in general, units would fail roughly with time by both causes of additive and independent damages.

This chapter considers three replacement policies that are combined with additive and independent damage, in which the unit is replaced at a planned time or when the total damage exceeds a failure level, whichever occurs first, and undergoes minimal repair when independent damage occurs. First, we take up a standard cumulative damage model where the unit suffers some damage due to shocks and the total damage is additive. However, it might be impossible to estimate and know occurrences of shocks and the total damage every at each shock. Second, the total damage is measured only at periodic times. Third, the total damage increases linearly with time approximately, and two continuous damage models whose total damage is distributed normally and exponentially are proposed. Expected cost rates of the above three models are obtained by using the techniques of cumulative processes in reliability theory, and optimal policies which minimize them are derived analytically and computed numerically.

3.2 Age Replacement Model

Suppose that shocks occur at a renewal process with a general distribution $F(t)$ with finite mean $1/\lambda$ and a density function $f(t) \equiv F'(t)$. An amount W_j of damage due to the jth shock has an identical distribution $G(x) \equiv \Pr\{W_j \leq x\}$ with finite mean μ, and the total damage is additive. We call it damage 1. It is assumed that the unit fails when the total damage exceeds a failure level K $(0 < K < \infty)$ at some shock, and it is replaced at a planed time T $(0 < T \leq \infty)$ or at failure, whichever occurs first. Then, the expected cost rate is, from (Nakagawa, 2007, p.42),

$$\widetilde{C}_1(T) = \frac{c_K - (c_K - c_T)\sum_{j=0}^{\infty}[F^{(j)}(T) - F^{(j+1)}(T)]G^{(j)}(K)}{\sum_{j=0}^{\infty} G^{(j)}(K)\int_0^T [F^{(j)}(t) - F^{(j+1)}(t)]\mathrm{d}t}, \tag{3.1}$$

where $\phi^{(j)}(x)$ $(j = 1, 2, \cdots)$ denotes the j-fold Stieltjes convolution of any function $\phi(x)$ with itself and $\phi^{(0)}(x) \equiv 1$ for $t \geq 0$, c_K = replacement cost at failure and

c_T = replacement cost at time T, where $c_K > c_T$.

Next, suppose that another damage 2 occurs at a nonhomogeneous Poisson process with an intensity function $h(t)$ and a mean-value function $H(t) \equiv \int_0^t h(u)\mathrm{d}u$, i.e., the probability of j occurrences of damage 2 during $(0, t]$ is

$$P_j(t) \equiv \frac{[H(t)]^j}{j!}\mathrm{e}^{-H(t)} \qquad (j = 0, 1, 2, \cdots).$$

It is assumed that damage 2 occurs independently of damage 1, and also its damage is not additive which is called independent damage (Nakagawa, 2007, p.21). That is, when damage 2 occurs, the unit undergoes only minimal repair. Thus, the expected number of occurrences of damage 2, i.e., minimal repairs, before the replacement is

$$\begin{aligned} N_M =& H(T)\sum_{j=0}^{\infty}[F^{(j)}(T) - F^{(j+1)}(T)]G^{(j)}(K) \\ &+ \sum_{j=0}^{\infty}[G^{(j)}(K) - G^{(j+1)}(K)]\int_0^T H(t)\mathrm{d}F^{(j+1)}(t) \\ =& \sum_{j=0}^{\infty}[G^{(j)}(K) - G^{(j+1)}(K)]\int_0^T [1 - F^{(j+1)}(t)]\mathrm{d}H(t) \\ =& \sum_{j=0}^{\infty} G^{(j)}(K)\int_0^T [F^{(j)}(t) - F^{(j+1)}(t)]\mathrm{d}H(t). \end{aligned} \tag{3.2}$$

Therefore, adding the minimal repair cost to $\widetilde{C}_1(T)$ in (3.1),

$$C_1(T) = \frac{\begin{array}{c} c_K - (c_K - c_T)\sum_{j=0}^{\infty} G^{(j)}(K)[F^{(j)}(T) - F^{(j+1)}(T)] \\ +c_M \sum_{j=0}^{\infty} G^{(j)}(K)\int_0^T [F^{(j)}(t) - F^{(j+1)}(t)]\mathrm{d}H(t) \end{array}}{\sum_{j=0}^{\infty} G^{(j)}(K)\int_0^T [F^{(j)}(t) - F^{(j+1)}(t)]\mathrm{d}t}, \tag{3.3}$$

where c_M = minimal repair cost for damage 2. Clearly,

$$C_1(0) \equiv \lim_{T\to 0} C_1(T) = \infty,$$

$$C_1(\infty) \equiv \lim_{T\to\infty} C_1(T) = \frac{c_K + c_M\sum_{j=0}^{\infty} G^{(j)}(K)\int_0^{\infty}[F^{(j)}(t) - F^{(j+1)}(t)]\mathrm{d}H(t)}{[1 + M(K)]/\lambda},$$

where $M(K) \equiv \sum_{j=1}^{\infty} G^{(j)}(K)$, and note that the denominator represents the mean time to replacement when the total damage exceeds a failure level K. Thus, there exists a positive T_1^* $(0 < T_1^* \leq \infty)$ which minimizes $C_1(T)$ in (3.3).

We find an optimal T_1^* which minimizes $C_1(T)$ in (3.3). Differentiating $C_1(T)$ with respect to T and setting it equal to zero,

$$(c_K - c_T)\left\{Q_1(T)\sum_{j=0}^{\infty} G^{(j)}(K)\int_0^T [F^{(j)}(t) - F^{(j+1)}(t)]\mathrm{d}t - \sum_{j=0}^{\infty} F^{(j+1)}(T)[G^{(j)}(K) - G^{(j+1)}(K)]\right\} + c_M \sum_{j=0}^{\infty} G^{(j)}(K)\int_0^T [F^{(j)}(t) - F^{(j+1)}(t)][h(T) - h(t)]\mathrm{d}t = c_T, \tag{3.4}$$

where

$$Q_1(T) \equiv \frac{\sum_{j=0}^{\infty} f^{(j+1)}(T)[G^{(j)}(K) - G^{(j+1)}(K)]}{\sum_{j=0}^{\infty} G^{(j)}(K)[F^{(j)}(T) - F^{(j+1)}(T)]}.$$

It can be clearly seen that if $Q_1(T)$ is strictly increasing and $h(t)$ is increasing, or $Q_1(T)$ is increasing and $h(t)$ is strictly increasing, then the left-hand side of (3.4) is strictly increasing from 0 to

$$c_M\left\{\frac{h(\infty)}{\lambda}[1 + M(K)] - \sum_{j=0}^{\infty} G^{(j)}(K)\int_0^{\infty} [F^{(j)}(t) - F^{(j+1)}(t)]h(t)\mathrm{d}t\right\} + (c_K - c_T)\left\{\frac{Q_1(\infty)}{\lambda}[1 + M(K)] - 1\right\}. \tag{3.5}$$

Thus, if (3.5) is greater than c_T, then there exists a finite and unique T_1^* $(0 < T_1^* < \infty)$ which satisfies (3.4). In this case, the expected cost rate is

$$C_1(T_1^*) = (c_K - c_T)Q_1(T_1^*) + c_M h(T_1^*). \tag{3.6}$$

Furthermore, let T_1 be a solution of equation

$$Q_1(T)\sum_{j=0}^{\infty} G^{(j)}(K)\int_0^T [F^{(j)}(t) - F^{(j+1)}(t)]\mathrm{d}t - \sum_{j=0}^{\infty} F^{(j+1)}(T)[G^{(j)}(K) - G^{(j+1)}(K)] = \frac{c_T}{c_K - c_T}, \tag{3.7}$$

then $T_1 > T_1^*$, and let T_2 be a solution of equation

$$\sum_{j=0}^{\infty} G^{(j)}(K)\int_0^T [F^{(j)}(t) - F^{(j+1)}(t)][h(T) - h(t)]\mathrm{d}t = \frac{c_T}{c_M}, \tag{3.8}$$

then $T_2 > T_1^*$. Both T_1 and T_2 would be useful for computing an optimal T_1^* as its upper bounds.

On the other hand, when $H(t) = \alpha t$, i.e., $h(t) = \alpha$ ($\alpha > 0$), from (3.5), if $Q_1(\infty)[1 + M(K)] > \lambda c_K/(c_K - c_T)$, then there exists a finite T_1^* ($0 < T_1^* < \infty$) which satisfies (3.4). In addition, when $F(t) = 1 - \mathrm{e}^{-\lambda t}$ and $G(x) = 1 - \mathrm{e}^{-x/\mu}$, it was shown in (Nakagawa, 2007, p.48) that $G^{(j)}(x) = \sum_{i=j}^{\infty}[(x/\mu)^i/i!]\mathrm{e}^{-x/\mu}$ and $M(x) = x/\mu$,

$$Q_1(T) = \frac{\lambda \sum_{j=0}^{\infty}[(\lambda T)^j/j!][G^{(j)}(K) - G^{(j+1)}(K)]}{\sum_{j=0}^{\infty}[(\lambda T)^j/j!]G^{(j)}(K)}$$

is strictly increasing from $\lambda \mathrm{e}^{-K/\mu}$ to λ. Thus, if $K/\mu > c_T/(c_K - c_T)$, then there exists a finite and unique T_1^* ($0 < T_1^* < \infty$) which satisfies (3.4).

In particular, when $K \to \infty$, the expected cost rate is, from (3.3),

$$C_1(T) = \frac{c_T + c_M H(T)}{T}, \tag{3.9}$$

which agrees with that of the standard periodic replacement (Nakagawa, 2005, p.102).

It might be better to do some maintenance only at each shock: The unit is replaced before time T when the total damage has exceeded a failure level K, and after T, it is replaced certainly at the next shock. Then, from (Nakagawa, 2007, p.55), the mean time to replacement is

$$\begin{aligned}E(L) =& \sum_{j=0}^{\infty}[G^{(j)}(K) - G^{(j+1)}(K)]\left\{\int_0^T \left[\int_{T-u}^{\infty}(t+u)\mathrm{d}F(t)\right]\mathrm{d}F^{(j)}(u)\right.\\&\left.+\int_0^T t\mathrm{d}F^{(j+1)}(t)\right\} + \sum_{j=0}^{\infty}G^{(j+1)}(K)\int_0^T\left[\int_{T-u}^{\infty}(t+u)\mathrm{d}F(t)\right]\mathrm{d}F^{(j)}(u)\\=&\frac{1}{\lambda}\sum_{j=0}^{\infty}G^{(j)}(K)F^{(j)}(T),\end{aligned}$$

and the expected number of occurrences of damage 2 before replacement is

$$N_M = \sum_{j=0}^{\infty}[G^{(j)}(K) - G^{(j+1)}(K)]\left\{\int_0^T \left[\int_{T-u}^{\infty}H(t+u)\mathrm{d}F(t)\right]\mathrm{d}F^{(j)}(u)\right.$$

$$+\int_0^T H(t)\mathrm{d}F^{(j+1)}(t)\Bigg\} + \sum_{j=0}^{\infty} G^{(j+1)}(K)\int_0^T \left[\int_{T-u}^{\infty} H(t+u)\mathrm{d}F(t)\right]\mathrm{d}F^{(j)}(u)$$

$$=\sum_{j=0}^{\infty} G^{(j)}(K)\int_0^T \left[\int_0^{\infty}[H(t+u)-H(u)]\mathrm{d}F(t)\right]\mathrm{d}F^{(j)}(u).$$

Therefore, the expected cost rate is

$$\widehat{C}_1(T) = \frac{\begin{array}{c} c_K - (c_K - c_T)\sum_{j=0}^{\infty} G^{(j+1)}(K)[F^{(j)}(T) - F^{(j+1)}(T)] \\ +c_M \sum_{j=0}^{\infty} G^{(j)}(K)\int_0^T \left[\int_0^{\infty}[H(t+u)-H(u)]\mathrm{d}F(t)\right]\mathrm{d}F^{(j)}(u) \end{array}}{\sum_{j=0}^{\infty} G^{(j)}(K)F^{(j)}(T)/\lambda}. \tag{3.10}$$

3.3 Periodic Replacement Model

It is assumed that each amount W_n $(n = 1, 2, \cdots)$ of damage due to shocks is measured only at periodic times nT_0 $(n = 1, 2, \cdots)$ for a given T_0 $(0 < T_0 < \infty)$ and has an identical distribution $G_T(x) \equiv \Pr\{W_n \leq x\}$ with mean μ_T. The other assumptions are the same as those in the age replacement model. Suppose that the unit is replaced at time NT_0 or at failure, whichever occurs first. Then, the expected cost rate is, from (Nakagawa, 2007, p.84),

$$\widetilde{C}_2(N) = \frac{c_K - (c_K - c_N)G_T^{(N)}(K)}{T_0\sum_{n=0}^{N-1} G_T^{(n)}(K)} \qquad (N = 1, 2, \cdots), \tag{3.11}$$

where c_N = replacement cost at time NT_0.

The expected number of occurrences of minimal repairs due to damage 2 is

$$N_M = \sum_{n=0}^{N-1} H[(n+1)T_0][G_T^{(n)}(K) - G_T^{(n+1)}(K)] + H(NT_0)G_T^{(N)}(K)$$
$$= \sum_{n=0}^{N-1}[H((n+1)T_0) - H(nT_0)]G_T^{(n)}(K). \tag{3.12}$$

Therefore, adding the minimal repair cost to $\widetilde{C}_2(N)$ in (3.11),

$$C_2(N) = \frac{\begin{array}{l} c_K - (c_K - c_N)G_T^{(N)}(K) \\ +c_M \sum_{n=0}^{N-1}[H((n+1)T_0) - H(nT_0)]G_T^{(n)}(K) \end{array}}{T_0\sum_{n=0}^{N-1} G_T^{(n)}(K)} \qquad (N = 1, 2, \cdots). \tag{3.13}$$

Clearly,

$$C_2(1) = \frac{c_K - (c_K - c_N)G_T(K) + c_M H(T_0)}{T_0},$$

$$C_2(\infty) \equiv \lim_{N\to\infty} C_2(N) = \frac{c_K + c_M \sum_{n=0}^{\infty}[H((n+1)T_0) - H(nT_0)]G_T^{(n)}(K)}{T_0[1 + M_T(K)]},$$

where $M_T(K) \equiv \sum_{n=1}^{\infty} G_T^{(n)}(K)$.

We find an optimal N_2^* which minimizes $C_2(N)$ in (3.13). From the inequality $C_2(N+1) - C_2(N) \geq 0$,

$$\begin{aligned}&(c_K - c_N)\left\{Q_2(N+1)\sum_{n=0}^{N-1} G_T^{(n)}(K) - [1 - G_T^{(N)}(K)]\right\}\\ &+ c_M\left\{[H((N+1)T_0) - H(NT_0)]\sum_{n=0}^{N-1} G_T^{(n)}(K)\right.\\ &\left. - \sum_{n=0}^{N-1}[H((n+1)T_0) - H(nT_0)]G_T^{(n)}(K)\right\} \geq c_N \qquad (N = 1, 2, \cdots),\end{aligned} \tag{3.14}$$

where

$$Q_2(N) \equiv \frac{G_T^{(N-1)}(K) - G_T^{(N)}(K)}{G_T^{(N-1)}(K)}.$$

Denote the let-hand side in (3.14) be $L_2(N)$,

$$\begin{aligned}L_2(N) - L_2(N-1) = &\sum_{n=0}^{N-1} G_T^{(n)}(K)\Big((c_K - c_N)[Q_2(N+1) - Q_2(N)]\\ &+ c_M\{[H((N+1)T_0) + H((N-1)T_0) - 2H(NT_0)]\}\Big).\end{aligned}$$

Therefore, if $Q_2(N)$ is strictly increasing and $h(t)$ is increasing, or $Q_2(N)$ is increasing and $h(t)$ is strictly increasing, then the left-hand side of (3.14) is strictly increasing to $L_2(\infty)$. Thus, if $L_2(\infty) > c_N$, then there exists a finite and unique minimum N_2^* $(1 \leq N_2^* < \infty)$ which satisfies (3.14). Furthermore, let N_1 be a solution of the equation

$$Q_2(N+1)\sum_{n=0}^{N-1} G_T^{(n)}(K) - [1 - G_T^{(N)}(K)] \geq \frac{c_N}{c_K - c_N}, \tag{3.15}$$

then $N_1 \geq N_2^*$, and let N_2 be a solution of the equation

$$[H((N+1)T_0) - H(NT_0)] \sum_{n=0}^{N-1} G_T^{(n)}(K) - \sum_{n=0}^{N-1} [H((n+1)T_0) - H(nT_0)] G_T^{(n)}(K) \geq \frac{c_N}{c_M}, \tag{3.16}$$

then $N_2 \geq N_2^*$.

When $G_T^{(j)}(x) = \sum_{i=j}^{\infty}[(x/\mu_T)^i/i!]\mathrm{e}^{-x/\mu_T}$,

$$Q_2(N) = \frac{(K/\mu_T)^{N-1}/(N-1)!}{\sum_{n=N-1}^{\infty}(K/\mu_T)^n/n!}$$

is strictly increasing from e^{-K/μ_T} to 1 (Nakagawa, 2007 p.24). Thus, if $K/\mu_T > c_N/(c_K - c_N)$, then there exist a finite and unique minimum N_1 which satisfies (3.15). On the other hand, when $H(t) = \alpha t$, if $Q_2(\infty)[1 + M_T(K)] > c_K/(c_K - c_N)$, then there exists a finite N_2^* $(1 \leq N_2^* < \infty)$ which satisfies (3.16). In addition, when $G(x) = 1 - \mathrm{e}^{-x/\mu_T}$, $N_2^* = N_1$.

In particular, when $K \to \infty$, the expected cost rate is

$$C_2(N) = \frac{c_N + c_M H(NT_0)}{NT_0}, \tag{3.17}$$

which agrees with that of the periodic replacement (Nakagawa, 2005, p.238).

3.4 Continuous Models

The hypothesis of the cumulative damage models discussed in the above two sections may be so strong that they are not so practicable for some applications: (1) It is more practical that the total damage stored within the unit will increase with time itself and fail eventually, but not be in a constant level until next shock occurs. (2) Referring to these applications of backup models or garbage collection models, due to the high frequency of computer processes in the modern society, it may be not so valid to assume that update of data or garbage occurrence follows an nonhomogeneous Poisson process, because the time intervals of events might be very short and unclear enough. In this section, we consider two continuous damage models where the total damage $Z(t)$ increases linearly with time t according two probabilistic laws (Nakagawa, 2007, p.26).

3.4.1 Model 1

It is assumed that the total amount of damage increases with t, i.e., the total damage at time t is $Z(t) = At$, where A is a random variable whose distribution is $W_A(x) \equiv \Pr\{A \le x\}$. Then, the probability that the unit does not fail in $(0, t]$ is

$$\Pr\{Z(t) \le K\} = \Pr\{At \le K\} = \Pr\{A \le K/t\} = W_A(K/t). \tag{3.18}$$

Suppose that the unit is replaced at time T or when the total damage exceeds K, whichever occurs first. Then, the mean time to replacement is

$$TW_A(K/T) + \int_0^T t\mathrm{d}[1 - W_A(K/t)] = \int_0^T W_A(K/t)\mathrm{d}t. \tag{3.19}$$

Thus, by the similar method of obtaining (3.3), the expected cost rate is

$$C_3(T) = \frac{c_K - (c_K - c_T)W_A(K/T) + c_M \int_0^T W_A(K/t)\mathrm{d}H(t)}{\int_0^T W_A(K/t)\mathrm{d}t}. \tag{3.20}$$

Let $r_A(t)$ be the failure rate of $W_A(t)$, i.e., $r_A(t) \equiv -W_A'(t)/W_A(t)$. Differentiating $C_3(T)$ with respect to T and setting it equal to zero,

$$\begin{aligned} &(c_K - c_T)\left\{r_A(K/T)\int_0^T W_A(K/t)\mathrm{d}t - [1 - W_A(K/T)]\right\} \\ &+ c_M \int_0^T W_A(K/t)[h(T) - h(t)]\mathrm{d}t = c_T. \end{aligned} \tag{3.21}$$

Thus, if $r_A(t)$ is strictly increasing and $h(t)$ is increasing or $r_A(t)$ is increasing and $h(t)$ is strictly increasing, if a solution T_3^* to (3.21) exists, it is unique.

Next, suppose that the unit is replaced at time NT_0 for $T_0 > 0$ and when the total damage exceeds K, whichever occurs first. Then, by the similar method of obtaining (3.20), the expected cost rate is

$$C_4(N) = \frac{\begin{array}{l} c_K - (c_K - c_N)W_A(K/NT_0) \\ +c_M \sum_{n=0}^{N-1}[H((n+1)T_0) - H(nT_0)]W_A(K/nT_0) \end{array}}{T_0 \sum_{n=0}^{N-1} W_A(K/nT_0)} \quad (N = 1, 2, \cdots), \tag{3.22}$$

where define that when $n = 0$, $W_A(K/nT_0) \equiv 1$. From the inequality $C_4(N+1) - C_4(N) \geq 0$,

$$(c_K - c_N)\left\{Q_4(N)\sum_{n=0}^{N-1} W_A(K/nT_0) - [1 - W_A(K/NT_0)]\right\}$$
$$+ c_M\left\{[H((N+1)T_0) - H(NT_0)]\sum_{n=0}^{N-1} W_A(K/nT_0)\right.$$
$$\left.- \sum_{n=0}^{N-1}[H((n+1)T_0) - H(nT_0)]W_A(K/nT_0)\right\} \geq c_N \qquad (N = 1, 2, \cdots), \qquad (3.23)$$

where

$$Q_4(N) \equiv \frac{W_A(K/NT_0) - W_A(K/(N+1)T_0)}{W_A(K/NT_0)}.$$

Thus, if $Q_4(N)$ is strictly increasing and $h(t)$ is increasing, or $Q_4(N)$ is increasing and $h(t)$ is strictly increasing, if a solution N_4^* to (3.23) exists, its minimum is unique.

3.4.2 Model 2

It is assumed that $Z(t) = \mu_A t + B_t$, where B_t has a probability distribution $\Pr\{B_t \leq x\} \equiv W_B(x)$, which is called a Brownian motion with drift (Ross, 1983, p.197). Then, the probability that the unit does not fail in $(0, t]$ is

$$\Pr\{Z(t) \leq K\} = \Pr\{B_t \leq K - \mu_A t\} = W_B(K - \mu_A t).$$

Thus, by replacing formally $W_A(K/t)$ in (3.20) with $W_B(K - \mu_A t)$, the expected cost rate is

$$C_5(T) = \frac{c_K - (c_K - c_T)W_B(K - \mu_A T) + c_M \int_0^T W_B(K - \mu_A t)\mathrm{d}H(t)}{\int_0^T W_B(K - \mu_A t)\mathrm{d}t}. \qquad (3.24)$$

Let $r_B(t)$ be the failure rate of $W_B(t)$, *i.e.*, $r_B(t) \equiv -W_B'(t)/W_B(t)$. Differentiating $C_5(T)$ with respect to T and setting it equal to zero,

$$(c_K - c_T)\left\{r_B(K - \mu_A T)\int_0^T W_B(K - \mu_A t)\mathrm{d}t - [1 - W_B(K - \mu_A T)]\right\}$$

$$+ c_M \int_0^T W_B(K - \mu_A t)[h(T) - h(t)]\mathrm{d}t = c_T. \tag{3.25}$$

Thus, if $r_B(t)$ is strictly increasing and $h(t)$ is increasing or $r_B(t)$ is increasing and $h(t)$ is strictly increasing, if a solution T_5^* to (3.25) exists, it is unique.

3.5 Numerical Examples

It is assumed that $F(t) = 1 - \mathrm{e}^{-\lambda t}$ and $H(t) = \alpha t^m$ $(m \geq 1)$, and $G(x) = 1 - \mathrm{e}^{-x/\mu}$ for the standard model, $G_T(x) = 1 - \mathrm{e}^{-x/\mu_T}$ for the periodic model, A has a normal distribution $N(\mu_A, \sigma^2/t)$ for Model 1, and $W_B(x) = 1 - \mathrm{e}^{-x/\sigma\sqrt{t}}$ for Model 2 of the continuous model.

Furthermore, we set that the mean and variance of the total damage at any time nT_0 are equal approximately for all models,

$$E\{Z(nT_0)\} = \lambda\mu nT_0 = n\mu_T = \mu_A nT_0,$$
$$V\{Z(nT_0)\} = 2\lambda\mu^2 nT_0 = n\mu_T^2 = \sigma^2 nT_0.$$

Thus, we have

$$\lambda = \frac{2}{T_0}, \qquad \mu = \frac{\mu_T}{2}, \qquad \mu_A = \frac{\mu_T}{T_0}, \qquad \sigma^2 = \frac{\mu_T^2}{T_0}.$$

Table 3.1: Optimal T_1^* and $C_1(T_1^*)/c_T$ when $T_0 = 1$, $\mu_T = 1$ and $K = 20$.

m	c_K/c_T	$c_M\alpha/c_T = 0.01$		$c_M\alpha/c_T = 0.05$		$c_M\alpha/c_T = 0.1$	
		T_1^*	$C_1(T_1^*)/c_T$	T_1^*	$C_1(T_1^*)/c_T$	T_1^*	$C_1(T_1^*)/c_T$
	5	12.81	0.0999	12.81	0.1399	12.81	0.1899
1.0	10	11.59	0.1085	11.59	0.1485	11.59	0.1985
	20	10.61	0.1166	10.61	0.1567	10.61	0.2067
	5	9.63	0.2033	4.55	0.4572	3.25	0.6526
2.0	10	9.33	0.2044	4.55	0.4572	3.25	0.6526
	20	8.97	0.2061	4.55	0.4572	3.25	0.6526

Table 3.2: Optimal N_2^* and $C_2(N_2^*)/c_N$ when $T_0 = 1$, $\mu_T = 1$ and $K = 20$.

m	c_K/c_N	$c_M\alpha/c_N = 0.01$		$c_M\alpha/c_N = 0.05$		$c_M\alpha/c_N = 0.1$	
		N_2^*	$C_2(N_2^*)/c_N$	N_2^*	$C_2(N_2^*)/c_N$	N_2^*	$C_2(N_2^*)/c_N$
1.0	5	13	0.0992	13	0.1392	13	0.1892
	10	12	0.1095	12	0.1495	12	0.1995
	20	10	0.1195	10	0.1595	10	0.2095
2.0	5	10	0.2020	4	0.4500	3	0.6333
	10	9	0.2032	4	0.4500	3	0.6333
	20	9	0.2055	4	0.4500	3	0.6333

When $T_0 = 1$ and $\mu_T = 1$, i.e., $\lambda = 2$, $\mu = 1/2$, $\mu_A = 1$, $\sigma^2 = 1$, $H(t) = \alpha t^m$. Tables 3.1-3.5 present the optimal T_1^*, N_2^*, T_3^*, N_4^*, T_5^* and their resulting expected cost rates $C_1(T_1^*)/c_T$, $C_2(N_2^*)/c_N$, $C_3(T_3^*)/c_T$, $C_4(N_4^*)/c_N$, $C_5(T_5^*)/c_T$, respectively, for m, $c_M\alpha/c_i$ and c_K/c_i $(i = T, N)$ when $T_0 = 1$, $\mu_T = 1$ and $K = 20$. These indicate that all optimal times and cost rates have similar tendencies for the same given parameters: (1) When c_K/c_T or c_K/c_N increases, optimal times decrease and cost rates increase. (2) When m increases, optimal times decrease and cost rates increase, however, they become more stable as m become larger. (3) When $c_M\alpha/c_T$ or $c_M\alpha/c_N$ increases, optimal times decrease and cost rate increase, however, when $m = 1$, optimal times are not changed.

Table 3.3: Optimal T_3^* and $C_3(T_3^*)/c_T$ when $T_0 = 1$, $\mu_T = 1$ and $K = 20$.

m	c_K/c_T	$c_M\alpha/c_T = 0.01$		$c_M\alpha/c_T = 0.05$		$c_M\alpha/c_T = 0.1$	
		T_3^*	$C_3(T_3^*)/c_T$	T_3^*	$C_3(T_3^*)/c_T$	T_3^*	$C_3(T_3^*)/c_T$
1.0	5	12.97	0.0958	12.97	0.1358	12.97	0.1858
	10	11.93	0.1018	11.93	0.1418	11.93	0.1918
	20	11.17	0.1071	11.17	0.1471	11.17	0.1971
2.0	5	9.85	0.2013	4.53	0.4522	3.21	0.6425
	10	9.71	0.2016	4.53	0.4522	3.21	0.6425
	20	9.51	0.2020	4.53	0.4522	3.21	0.6425

Table 3.4: Optimal N_4^* and $C_4(N_4^*)/c_N$ when $T_0 = 1$, $\mu_T = 1$ and $K = 20$.

m	c_K/c_N	$c_M\alpha/c_N = 0.01$		$c_M\alpha/c_N = 0.05$		$c_M\alpha/c_N = 0.1$	
		N_4^*	$C_4(N_4^*)/c_N$	N_4^*	$C_4(N_4^*)/c_N$	N_4^*	$C_4(N_4^*)/c_N$
1.0	5	13	0.0951	13	0.1351	13	0.1851
	10	12	0.1012	12	0.1412	12	0.1912
	20	11	0.1067	11	0.1467	11	0.1967
2.0	5	10	0.2003	5	0.4500	3	0.6333
	10	10	0.2007	5	0.4500	3	0.6333
	20	9	0.2014	5	0.4500	3	0.6333

Table 3.5: Optimal T_5^* and $C_5(T_5^*)/c_T$ when $T_0 = 1$, $\mu_T = 1$ and $K = 20$.

m	c_K/c_T	$c_M\alpha/c_T = 0.01$		$c_M\alpha/c_T = 0.05$		$c_M\alpha/c_T = 0.1$	
		T_5^*	$C_5(T_5^*)/c_T$	T_5^*	$C_5(T_5^*)/c_T$	T_5^*	$C_5(T_5^*)/c_T$
1.0	5	10.95	0.1283	10.95	0.1683	10.95	0.2183
	10	9.21	0.1487	9.21	0.1887	9.21	0.2387
	20	7.87	0.1708	7.87	0.2108	7.87	0.2608
2.0	5	8.41	0.2119	4.36	0.4391	3.01	0.6035
	10	7.64	0.2209	4.33	0.4398	3.01	0.6036
	20	6.88	0.2330	4.28	0.4410	3.01	0.6037

We could explain the optimal policies as following: (1) When the replacement cost c_K at failure increases, we should advance the time of replacement, that is, T_1^*, N_2^*, T_3^*, N_4^*, T_5^* should be decreased, in order to reduce the probability of failures. (2) When m and $c_M\alpha/c_T$ or $c_M\alpha/c_N$ increase, i.e., the frequency and cost of minimal repair increase, the replacement time should be advanced to reduce the total minimal repair cost. (3) When m and $c_M\alpha/c_T$ or $c_M\alpha/c_N$ are both or either very larger, the replacement cost c_K have no effect on the optimal time. (4) When $m = 1$, that is, minimal repair occurs at a homogeneous Poisson process, the minimal repair cost c_M also have no effect on the optimal time. (5) Compared to T_1^* with N_2^* and T_3^* with N_4^*, all results are almost the same. Thus, if we could not estimate the total damage at each shock, then it would be sufficient to estimate it at periodic times. (6) Clearly, T_5^* is less than T_3^* because $E\{Z(t)\} = \mu_A t$ for Model 1 is smaller than $E\{Z(t)\} = \mu_A t + \sigma\sqrt{t}$ for Model 2. Model 2 could be applied practically to the continuous damage model where the total damage usually increases linearly with time t, however, it might change randomly on some additional accidents.

3.6 Concluding Remarks

We have discussed three kinds of replacement policies which are combined with additive and independent damages. The expected cost rates are obtained by using the techniques of cumulative processes in reliability theory. Optimal policies are derived analytically, optimal T_1^*, N_2^*, T_3^*, N_4^*, T_5^* and the resulting cost rates $C_1(T_1^*)/c_T$, $C_2(N_2^*)/c_N$, $C_3(T_3^*)/c_T$, $C_4(N_4^*)/c_N$, $C_5(T_5^*)/c_T$ are computed and compared numerically. By setting the mean and variance of the total damage at any time nT_0 are equal approximately for all models, all optimal times and cost rates have similar tendencies for the same given parameters in numerical analyses. However, the minimal repair cost c_M may be a variable and some damage caused by damage 1 may be reduced by some maintenance using the opportunity time of minimal repair. The obtained results and used methods in this paper would be applied in practise by modifying them suitably, as referred in Section 3.4, the continuous models could be applied in garbage collection models. For an another example of the applications, we could discuss a reorganization model of structural database (William and Tuel, 1978; Goda and Kitsuregawa, 2006) by replacing shock by update, and damage 1 by structural deterioration and damage 2 by split data deterioration.

Chapter 4

Random Working Times

This chapter takes up three maintenance policies for an operating system which works at random times for jobs. Each job causes some damage to the system and these damage are additive, and the system fails when the total damage has exceeded a failure level K. First, preventive maintenance is made at the Nth completion of working time for the standard model, and the system fails with probability $p(x)$ when the total damage is x for the minimal repair model. Second, the system is maintained at the first completion of some working times over time T. Third, when a limit number N of working times are introduced, maintenance is made at a planned time T or at a damage level Z. Using techniques of cumulative damage models, expected cost rates are obtained and optimal maintenance policies are discussed analytically and computed numerically.

4.1 Introduction

Most system deteriorate with age and use, and eventually, fail from either or both causes in random environment. If their failure rates increase with age and use, it may be wise to make some suitable maintenance at periodic times or at a certain number of failures. The policy with two variables would be effective where units suffer great deterioration due to both age and use. For example, some parts of aircraft have to be maintained at a specified number of flights and at a planned time since the last major overhaul (Nakagawa, 2008, p.149). This could also be applied to the maintenance of some parts of large complex systems such as switching devices and parts of transportation equipment, computers and plants (Nakagawa, 2005, p.95).

Some units in offices and industries execute jobs or computer procedures successively. For such units, it would be impossible or impractical to maintain or replace them in a strict periodic fashion, because the sudden suspension of the job may suffer losses of production in different degrees if there is no sufficient preparation in advance (Barlow and Proschan 1965, p.72; Nakagawa, 2005, p.245). When a job has a variable working cycle or processing time, it would be better to do maintenance or replacement after the job is completed (Sugiura et al., 2004). A representative example is to maintain a database or to perform a backup of data when a transaction is processing its sequences of operations, because it is necessary to guarantee ACID (atomicity, consistency, isolation, durability) properties of database transactions, especially for a distributed transaction across a distributed database (Haerder and Reuter, 1983; Gray and Reuter, 1992; Lewis, et al., 2002). In addition, some schedules of jobs that have random processing times were summarized (Pinedo, 2002). The properties of replacement times between two successive failed units, when the unit is replaced at such random times, were investigated (Stadje, 2003). A comparative study between periodic and random replacement was done (Nakagawa, et al., 2011). Furthermore, such a notion of “random maintenance" was applied to a parallel system with random number of units (Nakagawa and Zhao, 2012).

On the other hand, in crack growth models (Scarf, et al., 1996; Hopp and Kuo, 1998; Sobczyk and Trebicki, 2003) for aircrafts, it has been well-known that the unit fails when the size of one crack in it exceeds a failure level or the total sizes of all cracks attain to its certain level. For examples, rivets are normally adopted for jointing skin to stringers, ribs in aircraft structures and fatigue cracks are known to initiate at rivet holes. Although one crack may be not dangerous because its force decreases when it enlarges, cracks at several following rivets may be dangerous because they can cause multi-site damage (MSD). The MSD has been defined as the simultaneous occurrence of many tiny fatigue cracks at multiple locations in the same structural element and has become recently a major issue of aging aircrafts since the Aloha Airlines affair in 1988 (Schijve, 1995). If such small fatigue cracks would be of enough scale and density, the aircraft structure could no longer have sufficient strength. In case of riveted lap joints, small cracks of following rivet link up and cause the widespread fatigue damage. We must avoid such fatigue damage as much as one can, because it might cause the catastrophic aircraft disaster.

This chapter considers an operating system which works at successive random times for jobs and its maintenance policies, using the cumulative process (Nakagawa, 2007) by replacing shock by work: Each work causes some damage to the system and these damage are additive, and the system fails when the total damage has exceeded a failure level K. Maintenance is made at the Nth completion of working time for the standard model and the system fails with probability $p(x)$ when the total damage is x at some completion of working times for the minimal repair model. It might be useless to maintain an operating system even when a planned time T comes and be wise to maintain it at the completion of the some working times. We sometimes want to use the system as long as possible. From such a viewpoint, we propose the overtime policy where the system is maintained at the first completion of the some working times over time T for the overtime model. When the cumulative damage models are applied to crack growth models, a limit number N of working times are introduced, and maintenance is made at a planned time T or at a damage level Z for the last model. Expected cost rates of each model are obtained and optimal maintenance policies are discussed analytically and computed numerically.

4.2 Nth Working Time

4.2.1 Standard Policy

It is assumed that X_j $(j = 1, 2, \cdots)$ is the working time of an operating system and is independent and has an identical distribution $F(t) \equiv \Pr\{X_j \leq t\}$ with finite mean $1/\lambda$. That is, the system works at a renewal process with its distribution $F(t)$. It is also assumed that each work of a job incurs some damage to the system and the total damage is additive, which is called a cumulative damage model (Nakagawa, 2007). That is, suppose that the jth work causes some damage to the system in the amount Y_j $(j = 1, 2, \cdots)$ according to an identical distribution $G(x) \equiv \Pr\{Y_j \le x\}$ with finite mean $1/\mu$. Then, the probability that j times of works are completed in $(0, t]$ is, from (Nakagawa, 2007, p.17),

$$\Pr\{N(t) = j\} = F^{(j)}(t) - F^{(j+1)}(t) \qquad (j = 0, 1, 2, \cdots),$$

and the distribution of the total damage $Z(t)$ at time t is

$$\Pr\{Z(t) \le x\} = \sum_{j=0}^{\infty} G^{(j)}(x) \left[F^{(j)}(t) - F^{(j+1)}(t)\right],$$

where $\Phi^{(j)}(t)$ denotes the j-fold Stieltjes convolution of $\Phi(t)$ with itself and $\Phi^{(0)}(t) \equiv 1$ for $t > 0$ for any function $\Phi(t)$, and $M(x) \equiv \sum_{j=1}^{\infty} G^{(j)}(x)$ which is the renewal function of $G(x)$.

The operating system fails when the total damage has exceeded a failure level K, and its failure is detected and maintenance is made at the completion of working time. As the preventive maintenance policy, the system is maintained before failure at Nth $(N = 1, 2, \cdots)$ working time. Then, the mean time to maintenance is

$$\begin{aligned} E(L) &= \sum_{j=0}^{N-1} \left[G^{(j)}(K) - G^{(j+1)}(K)\right] \int_0^{\infty} \left[1 - F^{(j+1)}(t)\right] \mathrm{d}t \\ &\quad + G^{(N)}(K) \int_0^{\infty} \left[1 - F^{(N)}(t)\right] \mathrm{d}t \\ &= \sum_{j=0}^{N-1} G^{(j)}(K) \int_0^{\infty} \left[F^{(j)}(t) - F^{(j+1)}(t)\right] \mathrm{d}t = \frac{1}{\lambda} \sum_{j=0}^{N-1} G^{(j)}(K). \end{aligned} \tag{4.1}$$

Furthermore, let c_K and c_N be the respective maintenance costs at failure and the Nth working time with $c_K > c_N$. Then, the expected cost rate is (Nakagawa, 2007, p.44)

$$\frac{C_1(N)}{\lambda} = \frac{c_K - (c_K - c_N) G^{(N)}(K)}{\sum_{j=0}^{N-1} G^{(j)}(K)} \qquad (N = 1, 2, \cdots). \tag{4.2}$$

We find an optimal number N_1^* which minimizes the expected cost rate $C_1(N)$ in (4.2). From the inequality $C_1(N+1) - C_1(N) \ge 0$,

$$L_1(N) \sum_{j=0}^{N-1} G^{(j)}(K) - \left[1 - G^{(N)}(K)\right] \ge \frac{c_N}{c_K - c_N} \qquad (N = 1, 2, \cdots), \tag{4.3}$$

where

$$L_1(N) \equiv \frac{G^{(N)}(K) - G^{(N+1)}(K)}{G^{(N)}(K)}.$$

If $L_1(N)$ increases strictly with N , then the left-hand side of (4.3) also increases strictly with N . Therefore, if $L_1(\infty)\big[1 + M(K)\big] > c_K/\big(c_K - c_N\big)$, then there exists a finite and unique minimum N_1^* $(1 \leq N_1^* < \infty)$ which satisfies (4.3), and the resulting cost rate is

$$\big(c_K - c_N\big)L(N_1^*) < \frac{C_1(N_1^*)}{\lambda} \leq \big(c_K - c_N\big)L(N_1^* + 1).$$

If $L_1(N)$ increases with N, then we have two inequalities:

$$L_1(N)\sum_{j=0}^{N-1} G^{(j)}(K) - \big[1 - G^{(N)}(K)\big] \geq L_1(N) - [1 - G(K)], \tag{4.4}$$

$$L_1(N)\sum_{j=0}^{N-1} G^{(j)}(K) - \big[1 - G^{(N)}(K)\big] \geq L_1(N)\sum_{j=0}^{\infty} G^{(j)}(K) - 1, \tag{4.5}$$

which are proved as follows: For (4.4),

$$\begin{aligned} &L_1(N)\sum_{j=1}^{N-1} G^{(j)}(K) - \sum_{j=1}^{N-1}\big[G^{(j)}(K) - G^{(j+1)}(K)\big] \\ &= \sum_{j=1}^{N-1} G^{(j)}(K)[L_1(N) - L_1(j)] \geq 0, \end{aligned}$$

and for (4.5),

$$\begin{aligned} &L_1(N)\sum_{j=N}^{\infty} G^{(j)}(K) - \sum_{j=N}^{\infty}\big[G^{(j)}(K) - G^{(j+1)}(K)\big] \\ &= \sum_{j=N}^{\infty} G^{(j)}(K)[L_1(N) - L_1(j)] \leq 0. \end{aligned}$$

Let $\widetilde{N}_1$ and $\widetilde{N}_2$ be the respective solutions of the equations

$$L_1(N) + G(K) \geq \frac{c_K}{c_K - c_N},$$

$$L_1(N)[1 + M(K)] \geq \frac{c_K}{c_K - c_N},$$

then $N_1^* \le \widetilde{N}_1$ and $N_1^* \le \widetilde{N}_2$. The upper bounds $\widetilde{N}_1$ and $\widetilde{N}_2$ would be useful for computing an optimal N_1^* approximately when N is small and large, respectively.

In particular, when $G(x) = 1 - \mathrm{e}^{-\mu x}$, i.e., $G^{(j)}(x) = \sum_{i=j}^{\infty}\big[(\mu x)^i/i!\big]\mathrm{e}^{-\mu x}$, from Example 2.2 (Nakagawa, 2007, p.24), $L_1(N) = [(\mu K)^N/N!]/\sum_{i=N}^{\infty}[(\mu K)^i/i!]$ which increases from $\mathrm{e}^{-\mu K}$ to 1. Thus, if $\mu K > c_N/\big(c_K - c_N\big)$, then there exists a finite and unique minimum N_1^* $(1 \le N_1^* < \infty)$ which satisfies (4.3).

Table 4.1 presents optimal N_1^* which satisfy (4.3) and $C_1(N_1^*)/(\lambda c_N)$ for different μK and c_K/c_N. Clearly, N_1^* increase with μK and decrease with c_K/c_N. That is, to control effectively a high cost suffered for failure, we must make the maintenance time earlier as a failure level K is lower or a failure maintenance cost c_K is higher.

Table 4.1: Optimal N_1^* and $C_1(N_1^*)/(\lambda c_N)$ for μK and c_K/c_N.

c_K/c_N	$\mu K = 10$		$\mu K = 15$		$\mu K = 20$	
	N_1^*	$C_1(N_1^*)/(\lambda c_N)$	N_1^*	$C_1(N_1^*)/(\lambda c_N)$	N_1^*	$C_1(N_1^*)/(\lambda c_N)$
2	8	0.1561	11	0.1009	15	0.0734
5	5	0.2129	8	0.1282	12	0.0892
10	4	0.2533	7	0.1455	11	0.0995
15	4	0.2827	7	0.1567	10	0.1047
20	3	0.2993	6	0.1637	9	0.1095

4.2.2 Minimal Repair

It is assumed that the system fails with probability $p(x)$ with $p(0) \equiv 0$ when the total damage becomes x at the completion of working times, and undergoes only minimal repairs at failures, i.e., the total damage remains undisturbed by any minimal repair. Suppose that the system is maintained at the Nth working time. Then, the expected number of minimal repairs before maintenance is

$$N_M = \sum_{j=1}^{N-1} \int_0^{\infty} p(x)\mathrm{d}G^{(j)}(x). \tag{4.6}$$

Thus, the expected cost rate is,

$$\frac{C_2(N)}{\lambda} = \frac{1}{N}\left[c_M \sum_{j=1}^{N-1} \int_0^{\infty} p(x)\mathrm{d}G^{(j)}(x) + c_N\right] \qquad (N = 1, 2, \cdots), \tag{4.7}$$

where c_M is the minimal repair cost at each failure.

We find an optimal number N_2^* which minimizes the expected cost rate $C_2(N)$ in (4.7). From the inequality $C_2(N+1) - C_2(N) \geq 0$,

$$\sum_{j=0}^{N-1}\left[\int_0^{\infty} p(x)\mathrm{d}G^{(N)}(x) - \int_0^{\infty} p(x)\mathrm{d}G^{(j)}(x)\right] \geq \frac{c_N}{c_M} \qquad (N = 1, 2, \cdots). \tag{4.8}$$

Thus, if $\int_0^{\infty} p(x)\mathrm{d}G^{(j)}(x)$ increases strictly with j, then the left-hand side of (4.8) also increases strictly. When $p(x) = 1 - \mathrm{e}^{-\beta x}$,

$$\int_0^{\infty}\left[1 - \mathrm{e}^{-\beta x}\right]\mathrm{d}G^{(j)}(x) = 1 - [G^*(\beta)]^j ,$$

where $G^*(\beta) \equiv \int_0^{\infty} \mathrm{e}^{-\beta x}\mathrm{d}G(x) < 1$, which represents the Laplace-Stieltjes transform of $G(x)$. Hence, (4.8) becomes

$$[1 - G^*(\beta)] \sum_{j=1}^{N} j\,[G^*(\beta)]^{j-1} \geq \frac{c_N}{c_M}, \tag{4.9}$$

whose left-hand side increases strictly from $1 - G^*(\beta)$ to $1/\left[1 - G^*(\beta)\right]$. Therefore, if $c_M > c_N\left[1 - G^*(\beta)\right]$, then there exists a finite and unique minimum N_2^* $(1 \leq N_2^* < \infty)$ which satisfies (4.9). Clearly, if $c_M \geq c_N/[1 - G^*(\beta)]$, then $N_2^* = 1$.

Next, it is assumed that a probability function $p(x)$ is the degenerate distribution, i.e., $p(x) = 0$ for $x < K$ and 1 for $x \geq K$. Then, the system fails certainly when the total damage has exceeded a failure level K and undergoes minimal repair. Then, the expected number of failures before maintenance is

$$N_F = \sum_{j=1}^{N-1}\left[1 - G^{(j)}(K)\right]. \tag{4.10}$$

Thus, the expected cost rate is,

$$\frac{\widetilde{C}_2(N)}{\lambda} = \frac{1}{N}\left[c_M \sum_{j=1}^{N-1}\left[1 - G^{(j)}(K)\right] + c_N\right] \qquad (N = 1, 2, \cdots). \tag{4.11}$$

From $\widetilde{C}_2(N+1) - \widetilde{C}_2(N) \geq 0$,

$$\sum_{j=0}^{N-1} \left[G^{(j)}(K) - G^{(N)}(K)\right] \geq \frac{c_N}{c_M}. \tag{4.12}$$

Because $G^{(j)}(K)$ decreases with j, the left-hand side of (4.12) increases from $1 - G(K)$ to $1+M(K)$. Therefore, if $c_M > c_N/[1+M(K)]$, then there exists a finite and unique minimum $\widetilde{N}_2^*$ $(1 \leq \widetilde{N}_2^* < \infty)$ which satisfies (4.12). If $c_M \geq c_N/[1 - G(K)]$, then $\widetilde{N}_2^* = 1$.

Table 4.2: Optimal N_2^* and $C_2(N_2^*)/(\lambda c_M)$ for $G^*(\beta)$ and c_N/c_M.

c_N/c_M	$G^*(\beta)=0.90$		$G^*(\beta)=0.95$		$G^*(\beta)=0.99$	
	N_2^*	$C_2(N_2^*)/(\lambda c_M)$	N_2^*	$C_2(N_2^*)/(\lambda c_M)$	N_2^*	$C_2(N_2^*)/(\lambda c_M)$
0.5	3	0.2663	5	0.1951	10	0.0938
1.0	5	0.3801	7	0.2809	15	0.1337
1.5	7	0.4689	9	0.3449	18	0.1639
2.0	8	0.5381	10	0.3975	21	0.1892
5.0	16	0.8033	19	0.6077	35	0.2956

Table 4.3: Optimal $\widetilde{N}_2^*$ and $\widetilde{C}_2(\widetilde{N}_2^*)/(\lambda c_M)$ for μK and c_N/c_M.

c_N/c_M	$\mu K=10$		$\mu K=15$		$\mu K=20$	
	$\widetilde{N}_2^*$	$\widetilde{C}_2(\widetilde{N}_2^*)/(\lambda c_M)$	$\widetilde{N}_2^*$	$\widetilde{C}_2(\widetilde{N}_2^*)/(\lambda c_M)$	$\widetilde{N}_2^*$	$\widetilde{C}_2(\widetilde{N}_2^*)/(\lambda c_M)$
0.5	6	0.0871	9	0.0567	13	0.0414
1.0	7	0.1550	10	0.1033	14	0.0764
1.5	7	0.2175	11	0.1463	15	0.1094
2.0	8	0.2734	12	0.1877	15	0.1406
5.0	10	0.5683	14	0.4047	18	0.3118

Table 4.2 presents optimal N_2^* and $C_2(N_2^*)/(\lambda c_M)$ for different $G^*(\beta)$ and c_K/c_M. It is shown that N_2^* increase with both $G^*(\beta)$ and c_K/c_M. Table 4.3 presents optimal $\widetilde{N}_2^*$ and $\widetilde{C}_2(\widetilde{N}_2^*)/(\lambda c_M)$ for different μK and c_K/c_M. This shows

similar tendencies to Table 4.2. From these two tables, we can know that the optimal maintenance times become longer as the minimal repair cost c_M is lower and the failure level K is larger.

4.3 Overtime Policy

It may be wasteful to maintain an operating system at planned times even if it is working. For example, when the system is functioning for jobs with a variable working cycle and processing time, it would be better to be maintained after it has completed the work and process. It is assumed that the system is maintained before time T $(0 \leq T \leq \infty)$ when the total damage has exceeded a failure level K, and after T, it is maintained at the first completion of some working times. Then, the mean time to maintenance is (Nakagawa, 2007, p.55),

$$\begin{aligned} E(L) =& \sum_{j=0}^{\infty}\Big[G^{(j)}(K) - G^{(j+1)}(K)\Big]\left\{\int_0^T \left[\int_{T-u}^{\infty}(t+u)\mathrm{d}F(t)\right]\mathrm{d}F^{(j)}(u)\right. \\ &\left. + \int_0^T t\mathrm{d}F^{(j+1)}(t)\right\} + \sum_{j=0}^{\infty}G^{(j+1)}(K)\int_0^T\left[\int_{T-u}^{\infty}(t+u)\mathrm{d}F(t)\right]\mathrm{d}F^{(j)}(u) \\ =& \frac{1}{\lambda}\sum_{j=0}^{\infty}G^{(j)}(K)F^{(j)}(T). \end{aligned} \tag{4.13}$$

Therefore, the expected cost rate is

$$\frac{C_3(T)}{\lambda} = \frac{c_K - \big(c_K - c_T\big)\sum_{j=0}^{\infty}G^{(j+1)}(K)\big[F^{(j)}(T) - F^{(j+1)}(T)\big]}{\sum_{j=0}^{\infty}G^{(j)}(K)F^{(j)}(T)}, \tag{4.14}$$

where c_T is the maintenance cost at the completion of working time after T.

We find an optimal time T_3^* which minimizes the expected cost rate $C_3(T)$ in (4.13) when $F(t) = 1 - \mathrm{e}^{-\lambda t}$ and $G(x) = 1 - \mathrm{e}^{-\mu x}$, i.e., $F^{(j)}(t) = \sum_{i=j}^{\infty}\big[(\lambda t)^i/i!\big]\mathrm{e}^{-\lambda t}$ and $G^{(j)}(x) = \sum_{i=j}^{\infty}\big[(\mu x)^i/i!\big]\mathrm{e}^{-\mu x}$. Then, differentiating $C_3(T)$ with respect to T and setting it equal to zero,

$$Q_3(T)\sum_{j=0}^{\infty}G^{(j)}(K)F^{(j)}(t) - \sum_{j=0}^{\infty}\frac{(\lambda T)^j}{j!}\mathrm{e}^{-\lambda T}\left[1 - G^{(j+1)}(K)\right] = \frac{c_T}{c_K - c_T}, \tag{4.15}$$

where

$$Q_3(T) \equiv \frac{\sum_{j=0}^{\infty}(\lambda T)^j/j!\big[(\mu K)^{j+1}/(j+1)!\big]/\mathrm{e}^{-\mu K}}{\sum_{j=0}^{\infty}(\lambda T)^j/j!G^{(j+1)}(K)}.$$

Because $Q_3(T)$ increases strictly with T from $\mu K/\big(\mathrm{e}^{\mu K}-1\big)$ to 1, the left-hand side of (4.15) also increases strictly from

$$D \equiv \frac{\mu K - 1 + \mathrm{e}^{-\mu K}}{\mathrm{e}^{\mu K}-1} \leq \frac{\mu K}{2}$$

to μK. Therefore, we have the following optimal policies:

1. If $D \geq c_T/\big(c_K - c_T\big)$, then $T_3^* = 0$, i.e., the system is maintained at the first completion of working time, and the expected cost rate is

$$\frac{C_1(0)}{\lambda} = c_K - \big(c_K - c_T\big)G(K).$$

2. If $D < c_T/\big(c_K - c_T\big) < \mu K$, then there exists a finite and unique T_3^* ($0 < T_3^* < \infty$) which satisfies (4.15), and the resulting cost rate is

$$\frac{C_3(T_3^*)}{\lambda} = \big(c_K - c_T\big)Q(T_3^*).$$

3. If $\mu K < c_T/\big(c_K - c_T\big)$, then, $T_3^* = \infty$, i.e., the system is maintained only at failure, and the resulting cost rate is

$$\frac{C_3(\infty)}{\lambda} = \frac{c_K}{1+M(K)}.$$

Table 4.4 presents optimal λT_3^* which satisfy (4.15) and $C_3(T_3^*)/(\lambda c_T)$ for different μK and c_K/c_T. It is shown that T_3^* have the same tendencies with N_1^* for the same parameters and $\lambda T_3^* \approx N_1^*$ in Table 4.1. That is, optimal number and time are almost the same for two polices, however, from the economical point, the policy made at Nth working time is better than that at time T. However, from the convenient point, the policy at time T would be easier than that at number N, because it is not necessary to count the number of working times.

Table 4.4: Optimal λT_3^* and $C_3(T_3^*)/(\lambda c_T)$ for μK and c_K/c_T.

c_K/c_T	$\mu K = 10$		$\mu K = 15$		$\mu K = 20$	
	λT_3^*	$C_3(T_3^*)/(\lambda c_T)$	λT_3^*	$C_3(T_3^*)/(\lambda c_T)$	λT_3^*	$C_3(T_3^*)/(\lambda c_T)$
2	8.89	0.1667	12.21	0.1087	15.80	0.0795
5	4.71	0.2528	7.73	0.1512	10.86	0.1048
10	3.39	0.3179	6.09	0.1808	9.04	0.1216
15	2.83	0.3588	5.39	0.1986	8.22	0.1314
20	2.51	0.3894	4.98	0.2117	7.73	0.1386

4.4 Limit Number of Working Times

4.4.1 Expected Cost Rate

Some systems fail when the size of one crack in them exceeds a failure level or the total sizes of all cracks attain to its certain level, as introduced in crack growth models for aircrafts in Chapter 1. This section takes up the maintenance model where the system fails when the total damage has exceeded a failure level K or the total number of working time reaches to N ($N = 1, 2, \cdots$). As preventive maintenance, the system is maintained before failure at a planned time T ($0 < T \le \infty$) or at a damage level Z ($0 < Z \le K$), whichever occurs first. Then, the probability that the system is maintained at time T is

$$\mathrm{P}_T = \sum_{j=0}^{N-1} [F^{(j)}(T) - F^{(j+1)}(T)] G^{(j)}(Z), \tag{4.16}$$

the probability that the system is maintained at damage Z is

$$\mathrm{P}_Z = \sum_{j=0}^{N-1} F^{(j+1)}(T) \int_0^Z [G(K-x) - G(Z-x)] \, \mathrm{d}G^{(j)}(x), \tag{4.17}$$

and probability that the system is maintained at failure is

$$\mathrm{P}_F = F^{(N)}(T) G^{(N)}(Z) + \sum_{j=0}^{N-1} F^{(j+1)}(T) \int_0^Z \overline{G}(K-x) \mathrm{d}G^{(j)}(x). \tag{4.18}$$

Note that $P_T+P_Z+P_F \equiv 1$, and the mean time to maintenance is, from (Nakagawa, 2007, p.41),

$$E(L) = \sum_{j=0}^{N-1} G^{(j)}(Z) \int_0^T [F^{(j)}(t) - F^{(j+1)}(t)]\mathrm{d}t. \tag{4.19}$$

Therefore, the expected cost rate is

$$\frac{C_4(T,Z)}{\lambda} = \frac{\begin{array}{c} c_K - (c_K - c_T)\sum_{j=0}^{N-1}[F^{(j)}(T) - F^{(j+1)}(T)]G^{(j)}(Z) \\ -(c_K - c_Z)\sum_{j=0}^{N-1} F^{(j+1)}(T)\int_0^Z [G(K-x) - G(Z-x)]\,\mathrm{d}G^{(j)}(x) \end{array}}{\sum_{j=0}^{N-1} G^{(j)}(Z)\int_0^T [F^{(j)}(t) - F^{(j+1)}(t)]\mathrm{d}t}, \tag{4.20}$$

where c_T, c_Z, and c_K are the maintenance cost at time T, damage level Z, and at failure with $c_T < c_K$ and $c_Z < c_K$.

4.4.2 Optimal Planned Time

When the system is maintained only at time T before failure, the expected cost rate is, from (4.20),

$$C_4(T) \equiv \lim_{Z\to K} C_4(T,Z) = \frac{c_K - (c_K - c_T)\sum_{j=0}^{N-1}[F^{(j)}(T) - F^{(j+1)}(T)]G^{(j)}(K)}{\sum_{j=0}^{N-1} G^{(j)}(K)\int_0^T [F^{(j)}(t) - F^{(j+1)}(t)]\mathrm{d}t}. \tag{4.21}$$

We find an optimal T_4^* which minimizes $C_4(T)$ in (4.21). In particular, when $N = 1$,

$$C_4(T) = \frac{c_K - (c_K - c_T)\overline{F}(T)}{\int_0^T \overline{F}(t)\mathrm{d}t}. \tag{4.22}$$

which corresponds to the expected cost rate of a standard age replacement policy (Nakagawa, 2005, p.72). It is assumed that the failure rate $h(t) \equiv f(t)/\overline{F}(t)$ increases strictly and $h(\infty) \equiv \lim_{t\to\infty} h(t)$. Then, from (Nakagawa, 2005, p.73), if $h(\infty) > \lambda c_K/(c_K - c_T)$, then there exists a finite and unique T_4^* $(0 < T_4^* < \infty)$ which satifies

$$h(T)\int_0^T \overline{F}(t)\mathrm{d}t + \overline{F}(T) = \frac{c_K}{c_K - c_T}, \tag{4.23}$$

and the resulting cost rate is

$$C_4(T_4^*) = (c_K - c_T)h(T_4^*). \tag{4.24}$$

Note that if $F(t) = 1 - \mathrm{e}^{\lambda t}$, then $h(t) = \lambda$, and hence, $T_4^* = \infty$, i.e., the system is maintained after the first working time, and the expected cost rate is

$$C_4(\infty) = \lim_{T\to\infty} C_4(T) = \lambda c_K. \tag{4.25}$$

Clearly, from (4.21),

$$C_4(0) = \lim_{T\to 0} C_4(T) = \infty,$$
$$C_4(\infty) = \lim_{T\to\infty} C_4(T) = \frac{\lambda c_K}{\sum_{j=0}^{N-1} G^{(j)}(K)}.$$

Thus, there exists a positive T_4^* $(0 < T_4^* < \infty)$ which minimizes (4.21).

Differentiating $C_4(T)$ with respect to T and setting it equal to zero,

$$Q_4(T) \sum_{j=0}^{N-1} G^{(j)}(Z) \int_0^T [F^{(j)}(t) - F^{(j+1)}(t)]\mathrm{d}t + \sum_{j=0}^{N-1} [F^{(j)}(T) - F^{(j+1)}(T)] G^{(j)}(K) = \frac{c_K}{c_K - c_T}, \tag{4.26}$$

where

$$Q_4(T) \equiv \frac{-\sum_{j=0}^{N-1}[f^{(j)}(T) - f^{(j+1)}(T)]G^{(j)}(K)}{\sum_{j=0}^{N-1}[F^{(j)}(T) - F^{(j+1)}(T)]G^{(j)}(K)}.$$

Let $L_4(T)$ be the left-hand side of (4.26),

$$L_4(0) \equiv \lim_{T\to 0} L_4(T) = 1 < \frac{c_K}{c_K - c_T},$$
$$\frac{\mathrm{d}L_4(T)}{\mathrm{d}T} = \frac{\mathrm{d}Q_4(T)}{\mathrm{d}T} \sum_{j=0}^{N-1} G^{(j)}(K) \int_0^T [F^{(j)}(t) - F^{(j+1)}(t)]\mathrm{d}t.$$

Thus, if $Q_4(T)$ increases strictly, then $L_4(T)$ also increases strictly from 1 to

$$L_4(\infty) = \frac{Q_4(\infty)}{\lambda} \sum_{j=0}^{N-1} G^{(j)}(K).$$

Therefore, if $L_4(\infty) > c_K/(c_K - c_T)$, then there exits a finite and unique T_4^* $(0 < T_4^* < \infty)$ which satisfies (4.26), and the resulting cost rate is

$$C_4(T_4^*) = (c_K - c_T)Q_4(T_4^*). \tag{4.27}$$

In particular, when $F(t) = 1 - \mathrm{e}^{-\lambda t}$ for $N \geq 2$,

$$Q_4(T) = \lambda\left(1 - \frac{\sum_{j=0}^{N-2}[(\lambda T)^j/j!]G^{(j+1)}(K)}{\sum_{j=0}^{N-1}[(\lambda T)^j/j!]G^{(j)}(K)}\right). \tag{4.28}$$

Suppose that $G^{(j+1)}(K)/G^{(j)}(K)$ decreases strictly. In this case, (4.26) becomes

$$\frac{Q_4(T)}{\lambda}\sum_{j=0}^{N-1}G^{(j)}(K)\sum_{i=j+1}^{\infty}\frac{(\lambda T)^j}{j!}\mathrm{e}^{-\lambda T} + \sum_{j=0}^{N-1}\frac{(\lambda T)^j}{j!}\mathrm{e}^{-\lambda T}G^{(j)}(K) = \frac{c_K}{c_K - c_T}, \tag{4.29}$$

whose left-hand side increases strictly to $\sum_{j=0}^{N-1}G^{(j)}(K)$, because of, from Appendix, $Q_4(T)$ increases strictly to λ . Therefore, if $\sum_{j=0}^{N-1}G^{(j)}(K) > c_K/(c_K - c_T)$, then there exists a finite and unique T_4^* $(0 < T_4^* < \infty)$ which satisfies (4.29), and the resulting cost rate is given in (4.27).

If each damage is not additive, as introduced in Chapter 3, the expected cost rate in (4.21) is rewritten as

$$\widetilde{C}_4(T) = \frac{c_K - (c_K - c_T)\sum_{j=0}^{N-1}[F^{(j)}(T) - F^{(j+1)}(T)][G(K)]^j}{\sum_{j=0}^{N-1}[G(K)]^j\int_0^T[F^{(j)}(t) - F^{(j+1)}(t)]\mathrm{d}t}. \tag{4.30}$$

We can make similar discussions for deriving analytically an optimal policy which minimizes $\widetilde{C}_4(T)$.

4.4.3 Optimal Damage Level

When the system is maintained only at damage level Z before failure, the expected cost rate is, from (4.20),

$$\frac{C_4(Z)}{\lambda} = \frac{c_K - (c_K - c_Z)\sum_{j=0}^{N-1}\int_0^Z[G(K-x) - G(Z-x)]\,\mathrm{d}G^{(j)}(x)}{\sum_{j=0}^{N-1}G^{(j)}(Z)}, \tag{4.31}$$

We find an optimal level Z_4^* which minimizes the expected cost rate $C_4(Z)$ in (4.31). In particular, when $N = 1$, $C_4(Z)$ increases with Z, and hence, $Z_4^* = 0$. When $N = \infty$,

$$\frac{C_4(Z)}{\lambda} = \frac{c_K - (c_K - c_Z)\int_0^Z [G(K-x) - G(Z-x)]\mathrm{d}M(x)}{M(Z)}, \tag{4.32}$$

where $M(x) \equiv \sum_{j=0}^{\infty} G^{(j)}(x)$. If $M(K) > c_Z/(c_K - c_Z)$, then there exists a finite and unique Z_4^* $(0 < Z_4^* < K)$ which satisfies

$$\int_{K-Z}^{K} M(K-x)\mathrm{d}G(x) = \frac{c_Z}{c_K - c_Z}. \tag{4.33}$$

Denote $g(x)$ be a density function of $G(x)$. Differentiating $C_4(Z)$ in (4.31) with respect to Z for $N \geq 2$ and setting it equal to zero,

$$Q_4(Z)\sum_{j=0}^{N-1} G^{(j)}(Z) + \sum_{j=0}^{N-1}\left[\int_0^Z [G(K-x) - G(Z-x)]\,\mathrm{d}G^{(j)}(x)\right] = \frac{c_K}{c_K - c_Z}, \tag{4.34}$$

where

$$Q_4(Z) \equiv \frac{g^{(N)}(Z)}{\sum_{j=1}^{N-1} g^{(j)}(Z)} + \overline{G}(K-Z).$$

Let $L_4(Z)$ be the left-hand side of (4.34),

$$L_4(0) = Q_4(0), \qquad L_4(K) = Q_4(K)\sum_{j=0}^{N-1} G^{(j)}(K),$$
$$L_4'(Z) = Q_4'(Z)\sum_{j=0}^{N-1} G^{(j)}(Z).$$

If $Q_4(Z)$ increases strictly with Z, then $L_4(Z)$ also increases from $Q_4(0)$ to $L_4(K)$. In this case, we have the following optimal policies:

1. If $Q_4(0) \geq c_K/(c_K - c_Z)$, then $Z_4^* = 0$ and the resulting cost rate is $C_4(0)/\lambda = c_K$.

2. If $Q_4(0) < c_K/(c_K - c_Z) < Q_4(K)\sum_{j=0}^{N-1} G^{(j)}(K)$, then there exists a finite and unique Z_4^* $(0 < Z_4^* < K)$ which satisfies (4.34), and the resulting cost rate is

$$\frac{C_4(Z_4^*)}{\lambda} = (c_K - c_Z)Q_4(Z_4^*). \tag{4.35}$$

3. If $c_K/(c_K - c_Z) > Q_4(K)\sum_{j=0}^{N-1} G^{(j)}(K)$, then $Z_4^* = K$, and the resulting cost rate is

$$\frac{C_4(K)}{\lambda} = \frac{c_K}{\sum_{j=0}^{N-1} G^{(j)}(K)}. \tag{4.36}$$

In particular, when $K = \infty$, the expected cost rate is

$$\frac{C_4(Z)}{\lambda} = \frac{c_Z - (c_N - c_Z)G^{(N)}(Z)}{\sum_{j=0}^{N-1} G^{(j)}(Z)},$$

where c_N represents the maintenance cost when the total number of working times reaches to N. In this case, (4.34) is simplified as for $N \geq 2$,

$$\widetilde{Q}_4(Z)\sum_{j=0}^{N-1} G^{(j)}(Z) - G^{(N)}(Z) = \frac{c_Z}{c_K - c_Z},$$

where

$$\widetilde{Q}_4(Z) \equiv \frac{g^{(N)}(Z)}{\sum_{j=1}^{N-1} g^{(j)}(Z)}.$$

Furthermore, when $K = \infty$ and $N = \infty$, the expected cost rate is

$$\frac{C_4(Z)}{\lambda} = \frac{c_Z}{M(Z)},$$

and hence, $Z_4^* = \infty$.

When $G(x) = 1 - \mathrm{e}^{-\mu x}$ for $N \geq 2$,

$$Q_4(Z) = \frac{(\mu Z)^{N-1}/(N-1)!}{\sum_{j=0}^{N-2}[(\mu Z)^j/j!]} + \mathrm{e}^{-\mu(K-Z)},$$

and (4.34) becomes

$$\sum_{j=0}^{N-1}\left\{Q_4(Z)\sum_{i=j}^{\infty}\frac{(\mu Z)^i}{i!}\mathrm{e}^{-\mu Z} + \left[1 - \mathrm{e}^{-\mu(K-Z)}\right]\frac{(\mu Z)^j}{j!}\right\} = \frac{c_K}{c_K - c_Z},$$

and when $N = \infty$,

$$\mu Z \mathrm{e}^{-\mu(K-Z)} = \frac{c_Z}{c_K - c_Z}.$$

It can be easily seen that $Q_4(Z)$ increases with Z from $\mathrm{e}^{-\mu K}$ to $Q_4(K)$. Therefore, if $c_K/(c_K - c_Z) < Q_4(K)\sum_{j=0}^{N-1} G^{(j)}(K)$, then there exists a finite and unique Z_4^* $(0 < Z_4^* < K)$ which satisfies (4.34).

4.4.4 Numerical Examples

We compute numerically optimal T_4^* and Z_4^* which minimize $C_4(T)$ in (4.21) and $C_4(Z)$ in (4.31) for $N \geq 2$, respectively, when $F(t) = 1 - \mathrm{e}^{-\lambda t}$ and $G(x) = 1 - \mathrm{e}^{-\mu x}$. Then, an optimal T_4^* satisfies

$$\sum_{j=0}^{N-1} G^{(j)}(K) \sum_{i=j+1}^{\infty} \frac{(\lambda T)^i}{i!}\mathrm{e}^{-\lambda T}\left(1 - \frac{\sum_{j=0}^{N-2}[(\lambda T)^j/j!]G^{(j+1)}(K)}{\sum_{j=0}^{N-1}[(\lambda T)^j/j!]G^{(j)}(K)}\right) + \sum_{j=0}^{N-1}\frac{(\lambda T)^j}{j!}\mathrm{e}^{-\lambda T}G^{(j)}(K) = \frac{c_K}{c_K - c_T}, \tag{4.37}$$

and an optimal Z_4^* satisfies

$$\sum_{j=0}^{N-1} G^{(j)}(Z)\left(\frac{(\mu Z)^{N-1}/(N-1)!}{\sum_{j=0}^{N-2}[(\mu Z)^j/j!]} + \mathrm{e}^{-\mu(K-Z)}\right) + \sum_{j=0}^{N-1}\frac{(\mu Z)^j}{j!}(\mathrm{e}^{-\mu Z} - \mathrm{e}^{-\mu K}) = \frac{c_K}{c_K - c_Z}, \tag{4.38}$$

In particular, as $K \to \infty$, i.e., the system is maintained only when the number of working times reaches to N, (4.37) and (4.38) a rewritten as, respectively,

$$\frac{(\lambda T)^{N-1}/(N-1)!}{\sum_{j=0}^{N-1}[(\lambda T)^j/j!]}\left[N - \sum_{j=0}^{N-1}(N-j)\frac{(\lambda T)^j}{j!}\mathrm{e}^{-\lambda T}\right] + \sum_{j=0}^{N-1}\frac{(\lambda T)^j}{j!}\mathrm{e}^{-\lambda T} = \frac{c_K}{c_K - c_T}, \tag{4.39}$$

$$\frac{(\mu Z)^{N-1}/(N-1)!}{\sum_{j=0}^{N-2}[(\mu Z)^j/j!]}\left[N - \sum_{j=0}^{N-1}(N-j-1)\frac{(\mu Z)^j}{j!}\mathrm{e}^{-\mu Z}\right] + \sum_{j=0}^{N-1}\frac{(\mu Z)^j}{j!}\mathrm{e}^{-\mu Z} = \frac{c_K}{c_K - c_Z}. \tag{4.40}$$

Table 4.5 presents optimal λT_4^* and μZ_4^* for different N and μK when c_K/c_i $(i = T, Z)$. It shows that both λT_4^* and μZ_4^* increase with N and μK when $N \geq 5$, and are stable when $N = 2, 3, 4$ for different μK, because for small N, we must maintain the system at earlier times before the number of working times reaches to N. When $N = \infty$, it can be seen that $\mu Z_4^* > \lambda T_4^*$ for finite μK. When $K = \infty$, $\mu Z_4^* < \lambda T_4^*$ for finite N because the left-hand side of 4.39 is less than 4.40 for $\lambda T = \mu Z$ and $c_T = c_Z$. When $N = \infty$ and $\mu K = \infty$, then $\lambda T_4^* = \mu Z_4^* = \infty$, because $C_4(T) = c_T/T$ and $C_4(Z) = \lambda c_T/(1 + \mu Z)$.

Table 4.5: Optimal λT_4^* and μZ_4^* for μK and N.

N	$\mu K = 8$		$\mu K = 10$		$\mu K = \infty$	
	λT_4^*	μZ_4^*	λT_4^*	μZ_4^*	λT_4^*	μZ_4^*
2	1.305	0.226	1.305	0.226	1.305	0.226
3	1.512	0.779	1.512	0.779	1.512	0.779
4	1.956	1.371	1.956	1.374	1.957	1.375
5	2.462	1.973	2.472	1.984	2.476	1.986
6	2.986	2.576	3.021	2.606	3.037	2.611
7	3.489	3.161	3.580	3.236	3.628	3.249
8	3.934	3.699	4.124	3.867	4.242	3.899
9	4.284	4.154	4.624	4.486	4.874	4.560
10	4.513	4.498	5.050	5.068	5.522	5.232
20	4.695	5.004	5.750	6.709	12.549	12.373
∞	4.695	5.004	5.750	6.709	∞	∞

4.5 Concluding Remarks

We have discussed three maintenance policies for an operating system which works at successive random times for jobs, the system fails due to damage that can be additive caused by jobs. Using the technique of cumulative damage models, the expected cost rates have been obtained, and the optimal maintenance policies have been discussed analytically. Numerical examples have been computed for all models and some useful explanations have been given.

For the Nth working time model, we first discussed the standard model where maintenance is made at the Nth completion of working time, the model has been given in (Nakagawa, 2007), we discussed the approximate upper bounds and for computing the optimal policy when the number of working times are small and large, respectively. Minimal repair is a important policy when a large and complex system is operating, so we introduced such a repair to the above policy where the system fails with probability $p(x)$ when the total damage is x and undergoes minimal repair at failure. Exponential and degenerate distributions are applied to discuss the optimal minimal repair policies, respectively.

The overtime model is the simplest policy when we consider the random working times, that is, we can delay the maintenance time suitably in order to continue the works until they are finished. Compared with the standard model, it is interesting that the property of optimal polices for the standard model has similar tendencies with that for the overtime model, however, the resulting cost rates of the standard model is lower than that of the overtime model, because the overtime policies increase the probability of failures when they are delayed.

The limit number of working times model is modified from (Nakagawa, 2007, p.40), before discussion such a notion, we have introduced its applications in crack growth models for aircrafts. In this chapter, a limitation N is considered, and optimal operating time and damage level are discussed, it has been shown that two optimal policies have the same variation properties when the same parameters are given. As extended models, we can set the limitation is the operation time T or damage level Z, and obtain other optimal policies.

Appendix

Prove that

$$\widetilde{Q}_4(T) = \frac{\sum_{j=0}^{N-1}[(\lambda T)^j/j!]G^{(j+1)}(K)}{\sum_{j=0}^{N}[(\lambda T)^j/j!]G^{(j+1)}(K)} \tag{A.1}$$

decreases to 0 as $T \to \infty$. Differentiating $\widetilde{Q}_4(T)$ with respect to T,

$$\begin{aligned}\frac{\mathrm{d}\widetilde{Q}_4(T)}{\mathrm{d}T} =& \frac{1}{\{\sum_{j=0}^{N}[(\lambda T)^j/j!]G^{(j)}(K)\}^2}\left[\sum_{j=0}^{N-1} j\frac{(\lambda T)^{j-1}}{j!}G^{(j+1)}(K)\sum_{i=0}^{N}\frac{(\lambda T)^i}{i!}G^{(i)}(K)\right.\\ &\left. - \sum_{j=0}^{N-1}\frac{(\lambda T)^j}{j!}G^{(j+1)}(K)\sum_{i=0}^{N} i\frac{(\lambda T)^{i-1}}{i!}G^{(i)}(K)\right].\end{aligned}$$

The numerator is

$$\begin{aligned}&\sum_{j=0}^{N-1}\frac{(\lambda T)^j}{j!}G^{(j+1)}(K)\sum_{i=0}^{N}\frac{(\lambda T)^{i-1}}{i!}G^{(i)}(K)(j-i)\\ &= \sum_{i=0}^{N}\frac{(\lambda T)^{i-1}}{i!}G^{(i)}(K)\left[\sum_{j=0}^{i}\frac{(\lambda T)^j}{j!}G^{(j+1)}(K)(j-i) + \sum_{j=i}^{N-1}\frac{(\lambda T)^j}{j!}G^{(j+1)}(K)(j-i)\right]\\ &= \sum_{j=0}^{N}\frac{(\lambda T)^j}{j!}G^{(j+1)}(K)\sum_{i=j}^{N}\frac{(\lambda T)^{i-1}}{i!}G^{(i)}(K)(j-i)\\ &\quad + \sum_{j=0}^{N}\frac{(\lambda T)^{j-1}}{j!}G^{(j)}(K)\sum_{i=j}^{N}\frac{(\lambda T)^i}{i!}G^{(i+1)}(K)(i-j)\\ &< \sum_{j=0}^{N}\frac{(\lambda T)^{j-1}}{j!}G^{(j)}(K)\sum_{i=j}^{N}\frac{(\lambda T)^i}{i!}G^{(i)}(K)(i-j)\left[\frac{G^{(j+1)}(K)}{G^{(j)}(K)} - \frac{G^{(i+1)}(K)}{G^{(i)}(K)}\right].\end{aligned}$$

Thus, if $G^{(j+1)}(K)/G^{(j)}(K)$ decreases strictly, then $\widetilde{Q}_4(T)$ decreases strictly to 0, and hence, $Q_4(T)$ increases strictly to λ.

Chapter 5

Maintenance Last Policies

From the economical viewpoint of several combined PM policies in reliability theory, this chapter proposes a standard cumulative damage model in which the notion of maintenance last is applied, i.e., the unit undergoes preventive maintenances before failure at a planned time T, at a damage level Z, or at a shock number N, whichever occurs last. Expected cost rate is detailedly formulated, and optimal problems of two alternative policies which combined time–based with condition–based preventive maintenances are discussed, i.e., optimal T_L^* for N, Z_L^* for T, and N_L^* for T are rigorously obtained. Comparison methods between such a maintenance last and the conventional maintenance first are explored. It is determined theoretically and numerically which policy should be adopted, according to the different methods in different cases when the time–based PM policy is optimized or the condition–based PM polices are optimized.

5.1 Introduction

PM actions for damage models are generally grouped into time–based and condition–based maintenances: If we have no information on the condition of a unit, its maintenance should be done at some age or usage period, e.g., at a planned time T (Nakagawa, 2007, p.42). On the other hand, if we could monitor some selected measurement parameters such as fatigue, wear, crack, etc., its maintenance should be done based on such conditions by periodic or random inspections, e.g., at a shock number N or at a damage level Z (Nakagawa, 2007, p.44-45). Furthermore, it was shown from numerical examples (Nakagawa, 2007, p.52-53) that optimal PM policies based on shock and damage conditions show more superiority than that based on a planned time, i.e., from the viewpoint of expected maintenance cost rates, PM done at Z is better than those at T and N, and PM done at N is better than that at T in most cases.

It has been assumed in all policies until now that the unit is maintained preventively at some amount of quantities, e.g., age, operating period, usage number, damage level, etc., or at failure, whichever occurs first, which is called maintenance first (MF). These policies are reasonable in practical fields if a unique PM policy in which to prevent failure is performed. However, this is especially the case when several combined PM policies are done. Taking parts of an aircraft as an example, appropriate maintenances are usually scheduled at a total hours of operation or at a specified number of flights since the last major overhaul (Duchesne, Lawless, 2000). Maintenance models with two PM policies, such as age and usage number, age and failure number, etc., have been discussed (Nakagawa, 2008, p.149). However, it has be found that such MF models would cause frequent and unnecessary maintenances which may incur production losses, when two or more alternative PM policies are adopted (Zhao and Nakagawa, 2012).

In addition, it has been assumed in all PM policies that the catastrophic failure mode is supposed, i.e., the unit suffered for failures may incur heavy losses. If the maintenance cost after failure would be estimated to be not so high, then the unit should be operating as long as possible before failures, so that the notion of “whichever occurs last” (Chen, et al., 2010a, 2010b) was proposed, i.e., the unit is replaced preventively at a planned time or at a working number, whichever occurs last. To motivate such a newly proposed notion more clearly, the policies with “whichever occurs last and first” were defined as replacement last and first, and their optimization problems, comparison methods, and real applications were explored (Zhao and Nakagawa, 2012). Furthermore, such a replacement last was applied to a shock model with damage level Z (Zhao, et al., 2011a).

As the following studies of the notion "whichever occurs last", this chapter takes up a standard cumulative damage model in which maintenance last (ML) is applied, i.e., the unit undergoes PM before failure at a planned time T, at a damage level Z, or at a shock number N, whichever occurs last. We detailedly formulate the expected cost rate which has been given directly in (Zhao and Nakagawa, 2012). Optimal problems which combine time–based with condition–based PM policies are discussed, i.e., optimal T_L^* for N, Z_L^* for T, and N_L^* for T are rigorously obtained. To compare such results of ML with those of MF, optimal T_F^* for N, Z_F^* for T, and N_F^* for T are also derived by similar methods, and comparisons between optimal ML and MF policies are demonstrated in detail.

5.2 Expected Cost Rate

It is assumed that random variables X_j $(j = 1, 2, \cdots)$ are shock time intervals which are independent and have an identical distribution $F(t) \equiv \Pr\{X_j \le t\}$ with a finite mean $1/\lambda$ and a density function $f(t) \equiv \mathrm{d}F(t)/\mathrm{d}t$. Each shock causes a random amount of damage Y_j $(j = 1, 2, \cdots)$ to a unit according to an identical distribution $G(x) \equiv \Pr\{Y_j \le x\}$ with a finite mean $1/\mu$ and a density function $g(x) \equiv \mathrm{d}G(x)/\mathrm{d}x$, and these damages are additive. The unit fails when the total damage exceeds a threshold level K $(0 < K < \infty)$, its failure is immediately detected, and then corrective maintenance (CM) is done. Then, the probability that shocks occur j times in $[0, t]$ is (Nakagawa, 2007, p.17)

$$\Pr\{N(t) = j\} = F^{(j)}(t) - F^{(j+1)}(t) \qquad (j = 0, 1, 2, \cdots),$$

and the distribution of the total damage $Z(t)$ at time t is

$$\Pr\{Z(t) \le x\} = \sum_{j=0}^{\infty} G^{(j)}(x)[F^{(j)}(t) - F^{(j+1)}(t)],$$

where $\Phi^{(j)}(t)$ denotes the j–fold Stieltjes convolution of any function $\Phi(t)$ with itself and $\Phi^{(0)}(t) \equiv 1$ for $t > 0$.

As preventive maintenance (PM) policies, the unit is maintained before failure at a planned time T $(0 \le T \le \infty)$, at a damage level Z $(0 \le Z \le K)$, or at a shock number N ($N = 0, 1, 2, \cdots$), whichever occurs last, which is called maintenance last (ML). We assume that the unit becomes as good as new at each CM or PM, i.e., any maintenance is perfect. Then, the probability P_T that the unit is maintained at time T is

$$\mathrm{P}_T = \sum_{j=N}^{\infty} [F^{(j)}(T) - F^{(j+1)}(T)][G^{(j)}(K) - G^{(j)}(Z)], \tag{5.1}$$

the probability P_Z that it is maintained at damage Z is

$$\mathrm{P}_Z = \sum_{j=N}^{\infty} \overline{F}^{(j+1)}(T) \int_0^Z [G(K-x) - G(Z-x)] \mathrm{d}G^{(j)}(x), \tag{5.2}$$

where $\overline{\phi}(t) \equiv 1 - \phi(t)$, and the probability P_N that it is maintained at shock N is

$$\mathrm{P}_N = \overline{F}^{(N)}(T)[G^{(N)}(K) - G^{(N)}(Z)]. \tag{5.3}$$

The probability P_K that the unit is maintained at failure is divided into three cases: The probability that the total damage exceeds K at some shock after T and N is

$$\mathrm{P}_1 = \sum_{j=N}^{\infty} \overline{F}^{(j+1)}(T) \int_0^Z \overline{G}(K-x) \mathrm{d}G^{(j)}(x),$$

the probability that the total damage exceeds K at some shock after T when the total shock number is less than or equal to N is

$$\mathrm{P}_2 = \sum_{j=0}^{N-1} \overline{F}^{(j+1)}(T)[G^{(j)}(K) - G^{(j+1)}(K)],$$

and the probability that the total damage exceeds K at some shock before T is

$$\mathrm{P}_3 = \sum_{j=0}^{\infty} F^{(j+1)}(T)[G^{(j)}(K) - G^{(j+1)}(K)].$$

By summing up P_1, P_2, and P_3,

$$\mathrm{P}_K \equiv \mathrm{P}_1 + \mathrm{P}_2 + \mathrm{P}_3 = 1 - \sum_{j=N}^{\infty} \overline{F}^{(j+1)}(T) \int_Z^K \overline{G}(K-x) \mathrm{d}G^{(j)}(x), \tag{5.4}$$

where note that $\mathrm{P}_Z + \mathrm{P}_N + \mathrm{P}_T + \mathrm{P}_K \equiv 1$. Then, the mean time to maintenance is

$$\begin{aligned}
E(L) =& T \sum_{j=N}^{\infty} [F^{(j)}(T) - F^{(j+1)}(T)][G^{(j)}(K) - G^{(j)}(Z)] \\
&+ [G^{(N)}(K) - G^{(N)}(Z)] \int_T^{\infty} t \mathrm{d}F^{(N)}(t) \\
&+ \sum_{j=N}^{\infty} \int_T^{\infty} t \mathrm{d}F^{(j+1)}(t) \int_0^Z [G(K-x) - G(Z-x)] \mathrm{d}G^{(j)}(x) \\
&+ \sum_{j=N}^{\infty} \int_T^{\infty} t \mathrm{d}F^{(j+1)}(t) \int_0^Z \overline{G}(K-x) \mathrm{d}G^{(j)}(x) \\
&+ \sum_{j=0}^{N-1} [G^{(j)}(K) - G^{(j+1)}(K)] \int_T^{\infty} t \mathrm{d}F^{(j+1)}(t) \\
&+ \sum_{j=0}^{\infty} [G^{(j)}(K) - G^{(j+1)}(K)] \int_0^T t \mathrm{d}F^{(j+1)}(t) \\
=& \sum_{j=N}^{\infty} G^{(j)}(Z) \int_T^{\infty} [F^{(j)}(t) - F^{(j+1)}(t)] \mathrm{d}t \\
&+ \sum_{j=0}^{N-1} G^{(j)}(K) \int_T^{\infty} [F^{(j)}(t) - F^{(j+1)}(t)] \mathrm{d}t \\
&+ \sum_{j=0}^{\infty} G^{(j)}(K) \int_0^T [F^{(j)}(t) - F^{(j+1)}(t)] \mathrm{d}t. \qquad (5.5)
\end{aligned}$$

Therefore, from (5.4) and (5.5), the expected cost rate is

$$C_L(T,Z,N) = \frac{c_P + (c_F - c_P)[1 - \sum_{j=N}^{\infty} \overline{F}^{(j+1)}(T) \int_Z^K \overline{G}(K-x) \mathrm{d}G^{(j)}(x)]}{\begin{array}{c} \sum_{j=N}^{\infty} G^{(j)}(Z) \int_T^{\infty} [F^{(j)}(t) - F^{(j+1)}(t)] \mathrm{d}t \\ + \sum_{j=0}^{N-1} G^{(j)}(K) \int_T^{\infty} [F^{(j)}(t) - F^{(j+1)}(t)] \mathrm{d}t \\ + \sum_{j=0}^{\infty} G^{(j)}(K) \int_0^T [F^{(j)}(t) - F^{(j+1)}(t)] \mathrm{d}t \end{array}}, \qquad (5.6)$$

where c_P is the PM cost at T, Z, or N, and c_F is the CM cost at failure with $c_F > c_P$.

In the following sections we find optimal PM time T_L^*, damage Z_L^*, and shock N_L^*, which minimize $C_L(T) \equiv \lim_{Z \to 0} C_L(T,Z,N)$ for N, $C_L(Z) \equiv \lim_{N \to 0} C_L(T,Z,N)$ for T, and $C_L(N) \equiv \lim_{Z \to 0} C_L(T,Z,N)$ for T, respectively, and compare them with the conventional PM policies discussed in (Nakagawa, 2007, p.42-46), when $F(t) = 1 - \mathrm{e}^{-\lambda t}$, i.e., $F^{(j)}(t) = \sum_{i=j}^{\infty}[(\lambda t)^i/i!]\mathrm{e}^{-\lambda t}$ $(j = 0, 1, 2, \cdots)$. Clearly,

$$\lim_{T\to\infty} C_L(T) = \lim_{Z\to K} C_L(Z) = \lim_{N\to\infty} C_L(N) = \frac{\lambda c_F}{1 + M(K)}, \tag{5.7}$$

which is the expected cost rate when only CM is made, i.e., when the unit is maintained only at failure, where $M(K) \equiv \sum_{j=1}^{\infty} G^{(j)}(K)$ presents the expected number of shocks before the total damage exceeds a failure level K.

Furthermore, we define the following three policies, i.e., $T-$PM, $Z-$PM, and $N-$PM, as the standard PM policies in cumulative damage models:

$$C_S(T) \equiv \lim_{Z\to 0, N\to 0} C_L(T, Z, N) = \frac{c_F - (c_F - c_P)\sum_{j=0}^{\infty}[F^{(j)}(T) - F^{(j+1)}(T)]G^{(j)}(K)}{\sum_{j=0}^{\infty} F^{(j+1)}(T)G^{(j)}(K)\mathrm{d}t/\lambda}, \tag{5.8}$$

which is the expected cost rate when $T-$PM is made, i.e., when the unit is maintained preventively at a planned time T,

$$C_S(Z) \equiv \lim_{T\to 0, N\to 0} C_L(T, Z, N) = \frac{c_F - (c_F - c_P)[G(K) - \int_0^Z \overline{G}(K - x)\mathrm{d}M(x)]}{[1 + M(Z)]/\lambda}, \tag{5.9}$$

which is the expected cost rate when $Z-$PM is made, i.e., when the unit is maintained preventively at a damage level Z, and

$$C_S(N) \equiv \lim_{T\to 0, Z\to 0} C_L(T, Z, N) = \frac{c_F - (c_F - c_P)G^{(N)}(K)}{\sum_{j=0}^{N-1} G^{(j)}(K)/\lambda}, \tag{5.10}$$

which is the expected cost rate when $N-$PM is made, i.e., when the unit is maintained preventively at a shock number N.

Optimal policies T^*, Z^*, and N^*, which minimize $C_S(T)$ in (5.8), $C_S(Z)$ in (5.9), and $C_S(N)$ in (5.10), have been discussed (Nakagawa, 2007, p.42-46). From the numerical examples (Nakagawa, 2007, p.52-53), it has been shown that the standard policy made at Z is better than those at T and N, and the standard policy made at N is better than that at T in most cases.

5.3 Optimal Planned Time

Suppose that the unit is maintained before failure at T ($0 \leq T \leq \infty$) or at N ($N = 0, 1, 2, \cdots$), whichever occurs last. Then, putting that $Z = 0$ in (5.6),

$$\frac{C_L(T)}{\lambda} = \frac{c_P + (c_F - c_P)\{1 - \sum_{j=N}^{\infty} \overline{F}^{(j+1)}(T)[G^{(j)}(K) - G^{(j+1)}(K)]\}}{\sum_{j=0}^{N-1} \overline{F}^{(j+1)}(T)G^{(j)}(K) + \sum_{j=0}^{\infty} F^{(j+1)}(T)G^{(j)}(K)}. \tag{5.11}$$

Differentiating $C_L(T)$ with respect to T for N and setting it equal to zero,

$$R_L(T,N)\left[\sum_{j=0}^{N-1} \overline{F}^{(j+1)}(T)G^{(j)}(K) + \sum_{j=0}^{\infty} F^{(j+1)}(T)G^{(j)}(K)\right]$$
$$+ \sum_{j=N}^{\infty} \overline{F}^{(j+1)}(T)[G^{(j)}(K) - G^{(j+1)}(K)] - 1 = \frac{c_P}{c_F - c_P}, \tag{5.12}$$

where

$$R_L(t,N) \equiv \frac{\sum_{j=N}^{\infty} f^{(j+1)}(t)[G^{(j)}(K) - G^{(j+1)}(K)]}{\sum_{j=N}^{\infty} f^{(j+1)}(t)G^{(j)}(K)},$$

and $f^{(j+1)}(t) \equiv [\lambda(\lambda t)^j/j!]\mathrm{e}^{-\lambda t}$. From Appendix 1, when $G^{(j)}(K) = \sum_{i=j}^{\infty}[(\mu K)^i/i!]$ $\mathrm{e}^{-\mu K}$, it is proved that $R_L(t,N)$ increases strictly with t from $1-G^{(N+1)}(K)/G^{(N)}(K)$ to 1. Denoting the left-hand side of (5.12) by $V_L(T,N)$ and $r_L(t,N) \equiv \mathrm{d}R_L(t,N)/\mathrm{d}t$,

$$\frac{\mathrm{d}V_L(T,N)}{\mathrm{d}T} = r_L(T,N)\left[\sum_{j=0}^{N-1} \overline{F}^{(j+1)}(T)G^{(j)}(K) + \sum_{j=0}^{\infty} F^{(j+1)}(T)G^{(j)}(K)\right] > 0, \tag{5.13}$$

which follows that $V_L(T,N)$ increases strictly with T from

$$V_L(0,N) = \frac{G^{(N)}(K) - G^{(N+1)}(K)}{G^{(N)}(K)} \sum_{j=0}^{N-1} G^{(j)}(K) - \overline{G}^{(N)}(K)$$

to $V_L(\infty, N) = M(K)$.

Therefore, if $V_L(0,N) < c_P/(c_F - c_P) < M(K)$, then there exists a finite and unique T_L^* ($0 < T_L^* < \infty$) that satisfies (5.12), and the resulting cost rate is

$$\frac{C_L(T_L^*)}{\lambda} = (c_F - c_P)R_L(T_L^*, N). \tag{5.14}$$

If $M(K) \leq c_P/(c_F - c_P)$, then $T_L^* = \infty$, and the resulting cost rate is given by (5.7). If $V_L(0,N) \geq c_P/(c_F - c_P)$, then $T_L^* = 0$, and the resulting cost rate is given by (5.10).

5.3.1 Comparison with Maintenance First

Suppose that the unit is maintained before failure at T $(0 < T \leq \infty)$ or at N $(N = 1, 2, \cdots)$, whichever occurs first, which is called maintenance first (MF). Then, the expected cost rate is, from (Nakagawa, 2007, p.42),

$$\frac{C_F(T)}{\lambda} = \frac{c_P + (c_F - c_P)\sum_{j=0}^{N-1} F^{(j+1)}(T)[G^{(j)}(K) - G^{(j+1)}(K)]}{\sum_{j=0}^{N-1} F^{(j+1)}(T)G^{(j)}(K)}. \tag{5.15}$$

By the similar method above, we discuss optimal T_F^* as follows: Differentiating $C_F(T)$ with respect to T for N and setting it equal to zero,

$$R_F(T,N)\sum_{j=0}^{N-1} F^{(j+1)}(T)G^{(j)}(K) - \sum_{j=0}^{N-1} F^{(j+1)}(T)[G^{(j)}(K) - G^{(j+1)}(K)] = \frac{c_P}{c_F - c_P}, \tag{5.16}$$

where

$$R_F(t,N) \equiv \frac{\sum_{j=0}^{N-1} f^{(j+1)}(t)[G^{(j)}(K) - G^{(j+1)}(K)]}{\sum_{j=0}^{N-1} f^{(j+1)}(t)G^{(j)}(K)}.$$

From Appendix 2, when $G^{(j)}(K) = \sum_{i=j}^{\infty}[(\mu K)^i/i!]\mathrm{e}^{-\mu K}$, it is approved that $R_F(t,N) = 1 - G(K)$ for $N = 1$ and increases strictly with t for $2 \leq N < \infty$ from $1 - G(K)$ to $1 - G^{(N+1)}(K)/G^{(N)}(K)$. Denoting the left-hand side of (5.16) by $V_F(T,N)$ and $r_F(t,N) \equiv \mathrm{d}R_F(t,N)/\mathrm{d}t$,

$$\frac{\mathrm{d}V_F(T,N)}{\mathrm{d}T} = r_F(T,N)\sum_{j=0}^{N-1} F^{(j+1)}(T)G^{(j)}(K) \geq 0, \tag{5.17}$$

which follows that $V_F(T,N)$ increases with T from 0 to

$$V_F(\infty,N) = \frac{G^{(N)}(K) - G^{(N+1)}(K)}{G^{(N)}(K)}\sum_{j=0}^{N-1} G^{(j)}(K) - \overline{G}^{(N)}(K).$$

Therefore, if $V_F(\infty,N) > c_P/(c_F - c_P)$, then there exists a finite and unique T_F^* $(0 < T_F^* < \infty)$ that satisfies (5.16), and the resulting cost rate is

$$\frac{C_F(T_F^*)}{\lambda} = (c_F - c_P)R_F(T_F^*, N). \tag{5.18}$$

If $V_F(\infty, N) \leq c_P/(c_F - c_P)$, then $T_F^* = \infty$, and the resulting cost rate is given by (5.10).

Let N^* $(1 \leq N^* < \infty)$ be an optimal shock number which minimizes $C_S(N)$ in (5.10). Then, N^* is a unique and minimum solution of the inequality, from (Nakagawa, 2007, p.45),

$$R(N) \sum_{j=0}^{N-1} G^{(j)}(K) - \overline{G}^{(N)}(K) \geq \frac{c_P}{c_F - c_P}, \tag{5.19}$$

where

$$R(j) \equiv \frac{G^{(j)}(K) - G^{(j+1)}(K)}{G^{(j)}(K)} \qquad (j = 0, 1, 2, \cdots).$$

Denoting the left-hand side of (5.19) by $V_S(N)$, it is easily shown that $V_S(N) = V_L(0, N) = V_F(\infty, N)$. Then, the following comparison results between ML and MF can be given:

1. If a predetermined $N < N^*$, then $T_F^* = \infty$, and $0 < T_L^* < \infty$ when $M(K) > c_P/(c_F - c_P)$, and $T_L^* = \infty$ when $M(K) \leq c_P/(c_F - c_P)$, i.e., ML should be adopted when $N < N^*$.

2. If a predetermined $N \geq N^*$, then $T_L^* = 0$, and $0 < T_F^* < \infty$ when $V_S(N) > c_P/(c_F - c_P)$ and $T_F^* = \infty$ when $V_S(N) \leq c_P/(c_F - c_P)$, i.e., MF should be adopted when $N \geq N^*$ and $V_S(N) > c_P/(c_F - c_P)$, and the standard $N-$PM should be adopted when $N \geq N^*$ and $V_S(N) \leq c_P/(c_F - c_P)$.

In addition, from Appendixes 3 and 4, both $R_L(t, N)$ and $R_F(t, N)$ increase strictly with N, that is, $V_L(T, N)$ in (5.12) and $V_F(T, N)$ in (5.16) increase with N. In other words, if finite T_L^* and T_F^* exist, they would decrease with N.

5.3.2 Numerical Example

Suppose that shock intervals X_j $(j = 1, 2, \cdots)$ have an identical distribution $F(t) = 1 - e^{-\lambda t}$, and random amount of damage Y_j $(j = 1, 2, \cdots)$ have an exponential distribution $G(x) = 1 - e^{-\mu x}$. Table 5.1 presents optimal T_L^*, T_F^*, and their cost rates $C_L(T_L^*)$ and $C_F(T_F^*)$ for N and $c_P/(c_F - c_P)$ when $\lambda = 1$, $\mu = 1$ and $K = 10$.

Table 5.1 indicates:

Table 5.1: Optimal T_L^*, T_F^*, and their cost rates when $\lambda = 1$, $\mu = 1$, $K = 10$.

$\frac{c_P}{c_F - c_P}$	$N=2$				$N=5$				$N=8$				
	T_L^*	$C_L(T_L^*)$	T_F^*	$C_F(T_F^*)$	T_L^*	$C_L(T_L^*)$	T_F^*	$C_F(T_F^*)$	T_L^*	$C_L(T_L^*)$	T_F^*	$C_F(T_F^*)$	N^*
0.1	4.00	0.034	∞	0.050	0.00	0.026	∞	0.026	0.00	0.041	4.75	0.033	5
0.2	5.20	0.055	∞	0.100	1.28	0.046	∞	0.046	0.00	0.054	6.84	0.052	6
0.3	6.07	0.073	∞	0.150	3.56	0.065	∞	0.066	0.00	0.067	9.49	0.066	6
0.4	6.78	0.089	∞	0.200	5.01	0.084	∞	0.086	0.00	0.080	14.09	0.079	7
0.5	7.42	0.104	∞	0.250	6.09	0.100	∞	0.106	0.00	0.093	28.35	0.092	7
0.6	8.00	0.118	∞	0.300	6.97	0.115	∞	0.126	0.00	0.106	∞	0.106	8
0.7	8.56	0.131	∞	0.350	7.75	0.129	∞	0.146	0.00	0.119	∞	0.119	8
0.8	9.09	0.144	∞	0.400	8.44	0.142	∞	0.166	0.00	0.131	∞	0.131	8
0.9	9.61	0.156	∞	0.450	9.07	0.155	∞	0.186	0.00	0.144	∞	0.144	8
1.0	10.12	0.168	∞	0.500	9.68	0.167	∞	0.206	2.62	0.157	∞	0.157	9

1. There exist three cases between T_L^* and T_F^* according to N^*: $0 < T_L^* < \infty$ and $T_F^* = \infty$ for $N < N^*$, $0 < T_F^* < \infty$ and $T_L^* = 0$ for $N > N^*$, and $T_L^* = 0$ and $T_F^* = \infty$ for $N = N^*$. That is, ML should be adopted for $N < N^*$, e.g., when $N = 2$ and $c_P/(c_F - c_P) = 0.1$, $C_L(T_L^*) = 0.034 < C_F(T_F^*) = 0.050$; MF should be adopted for $N > N^*$, e.g., when $N = 8$ and $c_P/(c_F - c_P) = 0.1$, $C_F(T_F^*) = 0.033 < C_L(T_L^*) = 0.041$; the standard $N-$PM should be adopted when $N = N^*$, e.g., when $N = 5$ and $c_P/(c_F - c_P) = 0.1$, $C_F(T_F^*) = C_L(T_L^*) = 0.026$.

2. When $0 < T_L^* < \infty$ or $0 < T_F^* < \infty$, both T_L^* and T_F^* increase with the maintenance cost ratio $c_P/(c_F - c_P)$, i.e., decrease with c_F/c_P. When c_F/c_P increases, PM should be advanced to prevent a higher CM cost. In other words, the unit can be operating for a longer time as c_F/c_P becomes smaller, e.g., when $N = 2$ and $c_P/(c_F - c_P) = 0.1, 0.5$, $T_L^* = 4.00, 7.42$; when $N = 8$ and $c_P/(c_F - c_P) = 0.1, 0.5$, $T_F^* = 0.033, 0.092$.

3. When a predetermined N becomes smaller, ML shows more superior cases than MF, not only because of comparison results between optimal cost rates in Table 1, but also because ML can let the unit work as longer as possible.

For example, $\mu K = 10$ means that the unit will be suffered failure around 10 shocks, when $N = 2$, PM is done at 2 shocks for MF, no matter whether c_F/c_P is large or small; however, PM would be done around from $\lambda T_L^* \approx 4$ to $\lambda T_L^* \approx 10$ shocks according to c_F/c_P. It seem more reasonable when ML is adopted for such cases to avoid unnecessary maintenances.

5.4 Optimal Damage Level

Suppose that the unit is maintained before failure at T $(0 \le T \le \infty)$ or at Z $(0 \le Z \le K)$, whichever occurs last (Zhao, et al., 2011a). Then, putting that $N = 0$ in (5.6),

$$\frac{C_L(Z)}{\lambda} = \frac{c_P + (c_F - c_P)[1 - \sum_{j=0}^{\infty} \overline{F}^{(j+1)}(T) \int_Z^K \overline{G}(K-x)\mathrm{d}G^{(j)}(x)]}{\sum_{j=0}^{\infty} \overline{F}^{(j+1)}(T) G^{(j)}(Z) + \sum_{j=0}^{\infty} F^{(j+1)}(T) G^{(j)}(K)}. \tag{5.20}$$

Differentiating $C_L(Z)$ with respect to Z for T and setting it equal to zero,

$$\sum_{j=0}^{\infty} \overline{F}^{(j+1)}(T) \int_Z^K [\overline{G}(K-x) - \overline{G}(K-Z)]\mathrm{d}G^{(j)}(x) + \overline{G}(K-Z) \sum_{j=0}^{\infty} G^{(j)}(K) - 1 = \frac{c_P}{c_F - c_P}. \tag{5.21}$$

Denoting $\mathrm{d}G(x)/\mathrm{d}x \equiv g(x)$ and the left-hand side of (5.21) by $V_L(Z,T)$,

$$\frac{\mathrm{d}V_L(Z,T)}{\mathrm{d}Z} = g(K-Z) \left[\sum_{j=0}^{\infty} \overline{F}^{(j+1)}(T) G^{(j)}(Z) + \sum_{j=0}^{\infty} F^{(j+1)}(T) G^{(j)}(K) \right] > 0, \tag{5.22}$$

which follows that $V_L(Z,T)$ increases strictly with Z from

$$\sum_{j=0}^{\infty} F^{(j+1)}(T) \int_0^K [\overline{G}(K) - \overline{G}(K-x)]\mathrm{d}G^{(j)}(x) < 0$$

to $M(K)$.

Therefore, if $M(K) > c_P/(c_F - c_P)$, then there exists a unique Z_L^* $(0 < Z_L^* < K)$ that satisfies (5.21), and the resulting cost rate is

$$\frac{C_L(Z_L^*)}{\lambda} = (c_F - c_P)\overline{G}(K - Z_L^*). \tag{5.23}$$

If $M(K) \le c_P/(c_F - c_P)$, then $Z_L^* = K$, and the resulting cost rate is given by (5.7).

5.4.1 Comparison with Maintenance First

Suppose that the unit is maintained before failure at T $(0 < T \leq \infty)$ or at Z $(0 < Z \leq K)$, whichever occurs first. Then, the expected cost rate is, from (Nakagawa, 2007, p.53),

$$\frac{C_F(Z)}{\lambda} = \frac{c_P + (c_F - c_P)\sum_{j=0}^{\infty} F^{(j+1)}(T)\int_0^Z \overline{G}(K-x)\mathrm{d}G^{(j)}(x)}{\sum_{j=0}^{\infty} F^{(j+1)}(T)G^{(j)}(Z)}. \tag{5.24}$$

Differentiating $C_F(Z)$ with respect to Z for T and setting it equal to zero,

$$\sum_{j=0}^{\infty} F^{(j+1)}(T))\int_0^Z [\overline{G}(K-Z) - \overline{G}(K-x)]\mathrm{d}G^{(j)}(x) = \frac{c_P}{c_F - c_P}. \tag{5.25}$$

Denoting the left-hand side of (5.25) by $V_F(Z,T)$,

$$\frac{\mathrm{d}V_F(Z,T)}{\mathrm{d}Z} = g(K-Z)\sum_{j=0}^{\infty} F^{(j+1)}(T)G^{(j)}(Z) > 0, \tag{5.26}$$

which follows that $V_F(Z,T)$ increases strictly with Z from 0 to

$$N(K) \equiv \sum_{j=1}^{\infty} F^{(j)}(T)G^{(j)}(K).$$

Therefore, if $N(K) > c_P/(c_F - c_P)$, then there exists a unique Z_F^* $(0 < Z_F^* < K)$ that satisfies (5.25), and the resulting cost rate is

$$\frac{C_F(Z_F^*)}{\lambda} = (c_F - c_P)\overline{G}(K - Z_F^*). \tag{5.27}$$

If $N(K) \leq c_P/(c_F - c_P)$, then $Z_F^* = K$, and the resulting cost rate is given by (5.8).

Because $M(K) \geq N(K)$, there exist both unique Z_L^* $(0 < Z_L^* < K)$ and Z_F^* $(0 < Z_F^* < K)$ which satisfy (5.21) and (5.25) when $N(K) > c_P/(c_F - c_P)$. Compare the left-hand side of (5.21) and (5.25) by denoting

$$A(Z) \equiv V_L(Z,T) - V_F(Z,T).$$

Then,

$$A(0) \equiv \lim_{Z\to 0} A(Z) = \sum_{j=0}^{\infty} F^{(j+1)}(T)\int_0^K [\overline{G}(K) - \overline{G}(K-x)]\mathrm{d}G^{(j)}(x) < 0,$$

$$A(K) \equiv \lim_{Z\to K} A(Z) = M(K) - N(K) = \sum_{j=1}^{\infty} \overline{F}^{(j)}(T)G^{(j)}(K) > 0.$$

From (5.22) and (5.26),

$$\frac{\mathrm{d}A(Z)}{\mathrm{d}Z} = g(K-Z)\left[\sum_{j=0}^{\infty} \overline{F}^{(j+1)}(T)G^{(j)}(Z) + \sum_{j=0}^{\infty} F^{(j+1)}(T)[G^{(j)}(K) - G^{(j)}(Z)]\right] > 0.$$

Thus, there exists a unique Z_A^* $(0 < Z_A^* < K)$ which satisfies $A(Z) = 0$.

From (5.25), denoting that

$$L(Z_A^*) \equiv F^{(j+1)}(T)\int_0^{Z_A^*} [\overline{G}(K - Z_A^*) - \overline{G}(K-x)]\mathrm{d}G^{(j)}(x). \tag{5.28}$$

Then, the following comparison results can be given:

1. If $L(Z_A^*) < c_P/(c_F - c_P)$, then $Z_L^* < Z_F^*$, and hence, from (5.23) and (5.27), $C_L(Z_L^*) < C_F(Z_F^*)$, i.e., ML should be adopted.

2. If $L(Z_A^*) > c_P/(c_F - c_P)$, then $Z_F^* < Z_L^*$, i.e., MF should be adopted.

3. If $L(Z_A^*) = c_P/(c_F - c_P)$, then ML is the same with MF.

In addition, $V_L(Z,T)$ in (5.21) decreases with T and $V_F(Z,T)$ in (5.25) increases with T, that is, optimal Z_L^* increases with T while Z_F^* decreases with T. It also can be easily found that Z_A^* increases with T, in other words, MF would show more superior cases than MF when a predetermined T becomes smaller.

Furthermore, let Z^* $(0 < Z^* < K)$ be an optimal damage level which minimizes $C_S(Z)$ in (5.9). Then, Z^* is a unique solution of the equation, from (Nakagawa, 2007, p.45),

$$\int_0^Z [\overline{G}(K-Z) - \overline{G}(K-x)]\mathrm{d}M(x) + \overline{G}(K-Z) - \overline{G}(K) = \frac{c_P}{c_F - c_P}, \tag{5.29}$$

and the resulting cost rate is

$$\frac{C_S(Z^*)}{\lambda} = (c_F - c_P)\overline{G}(K - Z^*). \tag{5.30}$$

Denoting the left-hand side of (5.29) by $V_S(Z)$ and $B_1(Z) \equiv V_S(Z) - V_L(Z,T)$, then $B_1(0) > 0$, $B_1(K) = 0$, and

$$\frac{\mathrm{d}B_1(Z)}{\mathrm{d}Z} = g(K-Z)\sum_{j=0}^{\infty} F^{(j+1)}(T)[G^{(j)}(Z) - G^{(j)}(K)] < 0. \tag{5.31}$$

Similarly, denoting $B_2(Z) \equiv V_S(Z) - V_F(Z,T)$, then $B_2(0) = 0$, $B_2(K) > 0$, and

$$\frac{\mathrm{d}B_2(Z)}{\mathrm{d}Z} = g(K-Z)\sum_{j=0}^{\infty} \overline{F}^{(j+1)}(T)G^{(j)}(Z) > 0, \tag{5.32}$$

which follows that $Z^* < Z_L^*$ and $Z^* < Z_F^*$, from (5.23), (5.27), and (5.30), $C_S(Z^*) < C_L(Z_L^*)$ and $C_S(Z^*) < C_F(Z_F^*)$, i.e., the standard $Z-$PM should be adopted.

5.4.2 Numerical Example

Table 5.2 presents optimal Z_L^* and Z_F^*, their cost rates $C_L(Z_L^*)$ and $C_F(Z_F^*)$, and Z_A^* and $L(Z_A^*)$ for T and $c_P/(c_F - c_P)$ when $\lambda = 1$, $\mu = 1$ and $K = 10$.

Table 5.2: Optimal Z_L^*, Z_F^*, and their cost rates when $\lambda = 1$, $\mu = 1$, $K = 10$.

$\frac{c_P}{c_F - c_P}$	$T=2$				$T=5$				$T=8$				
	Z_L^*	$C_L(Z_L^*)$	Z_F^*	$C_F(Z_F^*)$	Z_L^*	$C_L(Z_L^*)$	Z_F^*	$C_F(Z_F^*)$	Z_L^*	$C_L(Z_L^*)$	Z_F^*	$C_F(Z_F^*)$	Z^*
0.1	5.95	0.017	7.02	0.051	6.32	0.025	6.24	0.023	6.90	0.045	6.00	0.018	5.92
0.2	6.53	0.031	7.71	0.101	6.73	0.038	6.89	0.045	7.13	0.057	6.63	0.034	6.52
0.3	6.88	0.044	8.11	0.151	7.01	0.050	7.28	0.066	7.31	0.068	7.00	0.049	6.87
0.4	7.13	0.057	8.40	0.202	7.23	0.063	7.56	0.087	7.46	0.079	7.27	0.065	7.12
0.5	7.32	0.069	8.62	0.252	7.40	0.074	7.77	0.108	7.59	0.089	7.47	0.079	7.32
0.6	7.48	0.080	8.80	0.301	7.54	0.085	7.95	0.129	7.70	0.100	7.64	0.094	7.48
0.7	7.61	0.092	8.95	0.350	7.67	0.097	8.10	0.150	7.80	0.111	7.78	0.109	7.61
0.8	7.73	0.103	9.09	0.403	7.78	0.109	8.23	0.170	7.89	0.121	7.91	0.124	7.73
0.9	7.84	0.115	9.20	0.449	7.88	0.120	8.34	0.190	7.98	0.133	8.02	0.138	7.84
1.0	7.93	0.126	9.31	0.502	7.97	0.131	8.45	0.212	8.06	0.144	8.12	0.153	7.93
Z_A^*	3.74				6.43				7.85				
$L(Z_A^*)$	0.003				0.123				0.751				

Table 5.2 indicates:

1. There exist two cases between Z_L^* and Z_F^* according to $L(Z_A^*)$: When $L(Z_A^*) < c_P/(c_F - c_P)$, then $Z_L^* < Z_F^*$, i.e., ML should be adopted; when $L(Z_A^*) > c_P/(c_F - c_P)$, $Z_F^* < Z_L^*$, i.e., MF should be adopted. For example, when $T = 2$, $L(Z_A^*) = 0.003$ which is less than all given $c_P/(c_F - c_P)$, then $C_L(T_L^*) < C_F(T_F^*)$; when $T = 5$, then $L(Z_A^*) = 0.123 > c_P/(c_F - c_P) = 0.1$, then $C_F(T_F^*) = 0.023 < C_L(T_L^*) = 0.025$.

2. Z_L^* and Z_F^* increase with the maintenance cost ratio $c_P/(c_F - c_P)$, i.e., decrease with c_F/c_P. The reason can be found as the same as that in Table 1. For example, when $T = 2$ and $c_P/(c_F - c_P) = 0.1, 0.5$, $Z_L^* = 5.95, 7.32$ and $Z_F^* = 7.02, 8.62$.

3. Z_L^* increases with T from Z^* and Z_F^* decreases with T to Z^*, e.g., when $c_P/(c_F - c_P) = 0.1$ and $T = 2, 5$, $Z_L^* = 5.95, 6.32$ and $Z_F^* = 7.02, 6.24$. Z_L^* for T has different properties from T_L^* for N, nevertheless, it shows the same tendencies with Table 1 that when a predetermined T becomes smaller, ML shows more superior cases than MF from the viewpoint of both optimal cost rates and unnecessary maintenances avoidance. For example, $\mu K = 10$, when $T = 2$, $C_L(T_L^*) < C_F(T_F^*)$, and PM is done around $\lambda T = 2$ shocks for MF and around from $\mu Z_L^* \approx 6$ to $\mu Z_L^* \approx 8$ shocks for ML; when $T = 5$, $C_L(T_L^*) < C_F(T_F^*)$ for $0.2 \le c_P/(c_F - c_P) \le 1.0$ and $C_F(T_F^*) < C_L(T_L^*)$ for $c_P/(c_F - c_P) = 0.1$, and PM is done around $\lambda T = 5$ shocks for MF and around from $\mu Z_L^* \approx 6$ to $\mu Z_L^* \approx 8$ shocks for ML.

5.5 Optimal Shock Number

We discuss an optimal N_L^* for T to minimize $C_L(N)$ which is given by (5.11). Forming the inequality $C_L(N+1) - C_L(N) \ge 0$,

$$\sum_{j=N}^{\infty} \overline{F}^{(j+1)}(T) G^{(j)}(K)[R(j) - R(N)] + R(N) \sum_{j=0}^{\infty} G^{(j)}(K) - 1 \ge \frac{c_P}{c_F - c_P}, \quad (5.33)$$

where $R(j)$ $(j = 0, 1, 2, \cdots)$ is given in (5.19).

From (Nakagawa, 2007, p.24), $G^{(j+1)}(K)/G^{(j)}(K)$ decreases strictly with j when $G^{(j)}(K) = \sum_{i=j}^{\infty}[(\mu K)^i/i!]\mathrm{e}^{-\mu K}$, then $R(j)$ increases strictly with j from $\mathrm{e}^{-\mu K}$ to 1. Denoting the left-hand side of (5.33) by $V_L(N,T)$, then $V_L(N+1,T)-V_L(N,T)$ is

$$[R(N+1)-R(N)]\left[\sum_{j=N}^{\infty}F^{(j+1)}(T)G^{(j)}(K)+\sum_{j=0}^{N-1}G^{(j)}(K)\right]>0, \tag{5.34}$$

which follows that $V_L(N,T)$ increases strictly with N from

$$\sum_{j=0}^{\infty}F^{(j+1)}(T)\int_0^K[\overline{G}(K)-\overline{G}(K-x)]\mathrm{d}G^{(j)}(x)<0$$

to $M(K)$.

Therefore, if $M(K) > c_P/(c_F - c_P)$, then there exists a unique and minimum N_L^* $(1 \le N_L^* < \infty)$ which satisfies (5.33), and the resulting cost rate is

$$(c_F - c_P)R(N_L^*-1) < \frac{C_L(N_L^*)}{\lambda} \le (c_F - c_P)R(N_L^*). \tag{5.35}$$

If $M(K) \le c_P/(c_F - c_P)$, then $N_L^* = \infty$ and the resulting cost rate is given by (5.7).

5.5.1 Comparison with Maintenance First

An optimal N_F^* for T to minimize $C_F(N)$ in (5.15) is a unique and minimum solution of the inequality

$$\sum_{j=0}^{N-1}F^{(j+1)}(T)G^{(j)}(K)[R(N)-R(j)] \ge \frac{c_P}{c_F - c_P}, \tag{5.36}$$

Denoting the left-hand side of (5.36) by $V_F(N,T)$, then $V_F(N+1,T)-V_F(N,T)$ is

$$[R(N+1)-R(N)]\sum_{j=0}^{N-1}F^{(j+1)}(T)G^{(j)}(K)>0, \tag{5.37}$$

which follows that $V_F(N,T)$ increases strictly with N to $N(K)$.

Therefore, if $N(K) > c_P/(c_F - c_P)$, then there exists a unique and minimum N_F^* $(1 \le N_F^* < \infty)$ which satisfies (5.36), and the resulting cost rate is

$$(c_F - c_P)R(N_F^*-1) < \frac{C_F(N_F^*)}{\lambda} \le (c_F - c_P)R(N_F^*). \tag{5.38}$$

If $N(K) \leq c_P/(c_F - c_P)$, then $N_F^* = \infty$ and the resulting cost rate is given by (5.8).

There exist both unique N_L^* $(1 \leq N_L^* < \infty)$ and N_F^* $(1 \leq N_F^* < \infty)$ which satisfy (5.33) and (5.36) when $N(K) > c_P/(c_F - c_P)$. Compare the left-hand side of (5.33) and (5.36) by denoting

$$A(N) \equiv V_L(N, T) - V_F(N, T).$$

From (5.34) and (5.37), $A(N+1) - A(N)$ is

$$[R(N+1) - R(N)] \left[\sum_{j=0}^{N-1} \overline{F}^{(j+1)}(T) G^{(j)}(K) + \sum_{j=N}^{\infty} F^{(j+1)}(T) G^{(j)}(K) \right] > 0,$$

which follows that $A(N)$ increases with N strictly to

$$A(\infty) = \sum_{j=0}^{\infty} \overline{F}^{(j)}(T) G^{(j)}(K) > 0.$$

Thus, there exists a unique and minimum N_A^* $(1 \leq N_A^* < \infty)$ which satisfies $A(N) \geq 0$.

From (5.36), denoting that

$$L(N_A^*) \equiv \sum_{j=0}^{N_A^*-1} F^{(j+1)}(T) G^{(j)}(K) [R(N_A^*) - R(j)]. \tag{5.39}$$

Then, the following comparison results can be given:

1. If $L(N_A^*) < c_P/(c_F - c_P)$, then $N_L^* \leq N_F^*$, and $C_L(N_L^*) \leq C_F(N_F^*)$, i.e., ML should be adopted.

2. If $L(N_A^* - 1) > c_P/(c_F - c_P)$, then $N_F^* \leq N_L^*$, i.e., MF should be adopted.

3. If $L(N_A^* - 1) \leq c_P/(c_F - c_P) \leq L(N_A^*)$, then either ML or MF may be better than the other, or the same with each other.

In addition, similar properties can be found as those shown for Z_L^* and Z_F^*, that is, optimal N_L^* increases with T while N_F^* decreases with T, and MF would show more superior cases than MF when a predetermined T becomes smaller.

Furthermore, let N^* $(1 \le N^* < \infty)$ be an optimal shock number which minimizes $C_S(N)$ in (5.10). Then, N^* is a unique and minimum solution of the inequality (5.19), and the resulting cost rate is

$$(c_F - c_P)R(N^* - 1) < \frac{C_S(N^*)}{\lambda} \le (c_F - c_P)R(N^*). \tag{5.40}$$

Denoting the left-hand side of (5.19) by $V_S(N)$ and $B_1(N) \equiv V_S(N) - V_L(N, T)$, then $B_1(\infty) = 0$, and $B_1(N + 1) - B_1(N)$ is

$$-[R(N+1) - R(N)] \sum_{j=N}^{\infty} F^{(j+1)}(T)G^{(j)}(K) < 0. \tag{5.41}$$

Similarly, denoting $B_2(N) \equiv V_S(N) - V_F(N, T)$, then $B_2(1) = \overline{F}(T)[G^{(1)}(K) - G^{(2)}(K)]$, and $B_1(N + 1) - B_1(N)$ is

$$[R(N+1) - R(N)] \sum_{j=0}^{N-1} \overline{F}^{(j+1)}(T)G^{(j)}(K) > 0, \tag{5.42}$$

which follows that $N^* \le N_L^*$ and $N^* \le N_F^*$, then $C_S(N^*) \le C_L(N_L^*)$ and $C_S(N^*) \le C_F(N_F^*)$, i.e., the standard $N-$PM should be adopted.

5.5.2 Numerical Example

Table 5.3 presents optimal N_L^* and N_F^*, their cost rates $C_L(N_L^*)$ and $C_F(N_F^*)$, N_A^*, and $L(N_A^*)$ and $L(N_A^* - 1)$ for T and $c_P/(c_F - c_P)$ when $\lambda = 1$, $\mu = 1$ and $K = 10$.

Table 5.3 indicates:

1. There exist three cases between N_L^* and N_F^* according to $L(N_A^*-1)$ and $L(N_A^*)$: When $L(N_A^*) < c_P/(c_F-c_P)$, then $N_L^* \le N_F^*$, i.e., ML should be adopted, e.g., when $T = 2$, $L(N_A^*) = 0.012$ which is less than all given $c_P/(c_F - c_P)$, then $C_L(N_L^*) < C_F(N_F^*)$; when $L(N_A^* - 1) > c_P/(c_F - c_P)$, then $N_F^* \le N_L^*$, i.e., MF should be adopted, e.g., when $T = 5$, $L(N_A^*-1) = 0.142 > c_P/(c_F-c_P) = 0.1$, then $C_F(N_F^*) = 0.029 < C_L(T_L^*) = 0.032$; when $L(N_A^* - 1) \le c_P/(c_F - c_P) \le L(N_A^*)$, then $N_L^* = N_F^*$, and either ML or MF may be better than the other, e.g., when $T = 5$ and $c_P/(c_F - c_P) = 0.2$, then $N_L^* = N_F^* = 6$ but $C_L(N_L^*) = 0.048 < C_F(N_F^*) = 0.052$, and when $T = 8$ and $c_P/(c_F - c_P) = 0.6$, then

Table 5.3: Optimal N_L^*, N_F^*, and their cost rates when $\lambda = 1$, $\mu = 1$, $K = 10$.

$\frac{c_P}{c_F - c_P}$	$T=2$				$T=5$				$T=8$				
	N_L^*	$C_L(N_L^*)$	N_F^*	$C_F(N_F^*)$	N_L^*	$C_L(N_L^*)$	N_F^*	$C_F(N_F^*)$	N_L^*	$C_L(N_L^*)$	N_F^*	$C_F(N_F^*)$	N^*
0.1	5	0.026	6	0.052	5	0.032	5	0.029	6	0.049	5	0.026	5
0.2	6	0.045	7	0.102	6	0.048	6	0.052	6	0.063	6	0.049	6
0.3	6	0.062	9	0.152	6	0.064	7	0.074	7	0.075	6	0.064	6
0.4	7	0.077	10	0.202	7	0.078	7	0.095	7	0.088	7	0.080	7
0.5	7	0.091	11	0.252	7	0.093	8	0.116	7	0.100	7	0.096	7
0.6	8	0.106	12	0.302	8	0.106	8	0.137	8	0.112	8	0.111	8
0.7	8	0.119	14	0.352	8	0.119	9	0.157	8	0.124	8	0.126	8
0.8	8	0.131	15	0.403	8	0.132	9	0.178	8	0.136	8	0.141	8
0.9	8	0.144	17	0.453	9	0.145	10	0.198	9	0.148	9	0.156	8
1.0	9	0.156	19	0.503	9	0.156	10	0.219	9	0.159	9	0.169	9
N_A^*	3				6				8				
$L(N_A^*)$	0.012				0.270				0.821				
$L(N_A^*-1)$	0.003				0.142				0.547				

$N_L^* = N_F^* = 8$ but $C_F(N_F^*) = 0.111 < C_L(N_L^*) = 0.112$, it is interesting that such a case number will increase when T becomes larger.

2. N_L^* and N_F^* and their cost rates $C_L(N_L^*)$ and $C_F(N_F^*)$ show similar properties with those in Table 5.2 but different from those in Table 5.1. For example, N_L^* and N_F^* increase with the maintenance cost ratio $c_P/(c_F - c_P)$, N_L^* increases with T from N^* and N_F^* decreases with T to N^*. So that analyses could be obtained in a similar way, e.g., $\mu K = 10$, when $T = 2$, PM is done around $\lambda T = 2$ shocks for MF and at from $N_L^* = 5$ to $N_L^* = 9$ shocks for ML

3. Compare $C_L(i_L)$ $(i = T, Z, N)$ in Tables 5.1–5.3, ML done at Z_L^* is better than those done at T_L^* and N_L^*, and ML done at N_L^* is better than that done at T_L^* in most cases, when a predetermined T or N is in a moderate size, e.g., 2 and 5; however, when T or N is large enough, e.g., 8, both PM done at T_L^* and at Z_L^* are better than that done at N_L^*, and PM done at Z_L^* is better than that done at T_L^* in most cases. In other words, if we could monitor

the damage level at every shock time, then optimizing condition–based PM policies would be better than time–based PM policy. For MF, by comparing $C_F(i_F)$ $(i = T, Z, N)$, the time–based PM policy shows more superior cases, and the condition–based PM policies are in a position of weakness in most cases.

5.6 Concluding Remarks

We have discussed the standard cumulative damage models with maintenance last (ML) and maintenance first (MF), i.e., the unit undergoes preventive maintenances (PM) before failure at a planned time T, at a damage level Z, or at a shock number N, whichever occurs last and first. We have detailedly formulated the expected cost rate of ML given in (Zhao and Nakagawa, 2012) and optimized three sub–PM policies which combined time–based with condition–based polices, i.e., optimal T_L^* for N, Z_L^* for T, and N_L^* for T. We also have optimized three sub–PM policies of MF whose expected cost rates have been given in (Nakagawa, 2007) by similar methods, i.e., optimal T_F^* for N, Z_F^* for T, and N_F^* for T. Comparisons between optimal ML and MF policies have been demonstrated in detail. It has been determined theoretically which policy should be adopted, according to the different methods in different cases.

A representative example of such a cumulative damage model with maintenance last is to maintain a database or to perform a backup of data. We can consider its necessity and feasibility from the following viewpoints: (i) Normally, the database is maintained at periodic times such as day, week, month, etc. However, when a transaction is processing its sequences of operations, it is necessary to guarantee ACID (atomicity, consistency, isolation, durability) properties of database transactions (Haerder and Reuter, 1983; Gray and Reuter, 1992; Lewis, et al., 2002), so that it is not advisable to suspend any transaction when it is under busy state. That is, a strict periodic maintenance is not always effective, and sometimes a random maintenance policy is performed by considering the idle states of the system. (ii) Cumulative damage models have been successfully formulated the incremental processes of updated data in a database, such as differential backup and cumulative backup (Qian, et al., 1999, 2002a, 2002b, 2010; Nakamura, et al., 2003), which have become the most popular backup manners in the world. In other words, we can monitor the cumulative updated data at any time. (iii) Replacing shock with random idle states, damage with updated data, and failure level K with backup threshold, we can modify such theoretical models into maintaining database or performing backup polices by considering the fault recovery costs. Obviously, the catastrophic failure mode is not sensible because the backup threshold K can be determined to be not so high by the database management system (DBMS), especially when two or more combined backup schedules are performed.

Appendix

1. When $F^{(j)}(t) = \sum_{i=j}^{\infty}[(\lambda t)^i/i!]\mathrm{e}^{-\lambda t}$, i.e., $f^{(j+1)}(t) = [\lambda(\lambda t)^j/j!]\mathrm{e}^{-\lambda t}$, prove that $R_L(t, N)$ increases strictly with t from $1 - G^{(N+1)}(K)/G^{(N)}(K)$ to 1, i.e., prove that

$$1 - R_L(t, N) \equiv \frac{\sum_{j=N}^{\infty}[(\lambda t)^j/j!]G^{(j+1)}(K)}{\sum_{j=N}^{\infty}[(\lambda t)^j/j!]G^{(j)}(K)} \tag{A.1}$$

decreases strictly with t from $G^{(N+1)}(K)/G^{(N)}(K)$ to 0 when $N = 0, 1, 2, \cdots$. In particular, when $N = 0$, the process of the proof has been given in (Nakagawa, 2007, p.49). When $N = 1, 2, \cdots$, differentiating $1 - R_L(t, N)$ with respect to t,

$$\frac{\lambda A_1(t, N)}{[\sum_{j=N}^{\infty}[(\lambda t)^j/j!]G^{(j)}(K)]^2},$$

where

$$A_1(t, N) \equiv \sum_{i=N}^{\infty}\frac{(\lambda t)^i}{i!}G^{(i)}(K)\sum_{j=N-1}^{\infty}\frac{(\lambda t)^j}{j!}G^{(j+2)}(K) - \sum_{i=N}^{\infty}\frac{(\lambda t)^i}{i!}G^{(i+1)}(K)\sum_{j=N-1}^{\infty}\frac{(\lambda t)^j}{j!}G^{(j+1)}(K). \tag{A.2}$$

We denote $G_k \equiv G^{(k+1)}(K)/G^{(k)}(K)$, and G_k decreases strictly with k from $G(K)$ to 0 when $G^{(j)}(K) = \sum_{i=j}^{\infty}[(\mu K)^i/i!]\mathrm{e}^{-\mu K}$ (Nakagawa, 2007, p.24), and (A.2) is rewritten as

$$\begin{aligned}
A_1(t,N) =& \sum_{i=N}^{\infty}\frac{(\lambda t)^i}{i!}\sum_{j=N-1}^{\infty}\frac{(\lambda t)^j}{j!}G^{(i)}(K)G^{(j+1)}(K)(G_{j+1}-G_i)\\
=& \sum_{i=N}^{\infty}\frac{(\lambda t)^i}{i!}\sum_{j=N-1}^{i-1}\frac{(\lambda t)^j}{j!}G^{(i)}(K)G^{(j+1)}(K)(G_{j+1}-G_i)\\
&+\sum_{i=N}^{\infty}\frac{(\lambda t)^i}{i!}\sum_{j=i}^{\infty}\frac{(\lambda t)^j}{j!}G^{(i)}(K)G^{(j+1)}(K)(G_{j+1}-G_i)\\
=& \sum_{i=N}^{\infty}i\frac{(\lambda t)^i}{i!}\sum_{j=i}^{\infty}\frac{(\lambda t)^j}{(j+1)!}G^{(i)}(K)G^{(j+1)}(K)(G_i-G_{j+1})\\
&+\sum_{i=N}^{\infty}\frac{(\lambda t)^i}{i!}\sum_{j=i}^{\infty}(j+1)\frac{(\lambda t)^j}{(j+1)!}G^{(i)}(K)G^{(j+1)}(K)(G_{j+1}-G_i)\\
=& \sum_{i=N}^{\infty}\frac{(\lambda t)^i}{i!}\sum_{j=i}^{\infty}\frac{(\lambda t)^j}{(j+1)!}G^{(i)}(K)G^{(j+1)}(K)(G_{j+1}-G_i)(j-i+1)<0.
\end{aligned}$$

Furthermore,

$$\lim_{t\to 0}\frac{\sum_{j=N}^{\infty}[(\lambda t)^j/j!]G^{(j+1)}(K)}{\sum_{j=N}^{\infty}[(\lambda t)^j/j!]G^{(j)}(K)}=\frac{G^{(N+1)}(K)}{G^{(N)}(K)}.$$

For any $N_1 \geq N$,

$$\lim_{t\to\infty}\frac{\sum_{j=N}^{N_1}[(\lambda t)^j/j!]G^{(j+1)}(K)}{\sum_{j=N}^{N_1}[(\lambda t)^j/j!]G^{(j)}(K)}=\frac{G^{(N_1+1)}(K)}{G^{(N_1)}(K)}.$$

Because N_1 is arbitrary,

$$\lim_{t\to\infty}\frac{\sum_{j=N}^{\infty}[(\lambda t)^j/j!]G^{(j+1)}(K)}{\sum_{j=N}^{\infty}[(\lambda t)^j/j!]G^{(j)}(K)}=\lim_{N_1\to\infty}\frac{G^{(N_1+1)}(K)}{G^{(N_1)}(K)}=0.$$

Thus, $1-R_L(t,N)$ decreases with t from $G^{(N+1)}(K)/G^{(N)}(K)$ to 0, which completes the proof that $R_L(t,N)$ increases with t from $1-G^{(N+1)}(K)/G^{(N)}(K)$ to 1.

2. Prove that $R_F(t,N)$ increases with t from $1-G(K)$ to $1-G^{(N+1)}(K)/G^{(N)}(K)$, i.e., prove that

$$1-R_F(t,N)\equiv\frac{\sum_{j=0}^{N-1}[(\lambda t)^j/j!]G^{(j+1)}(K)}{\sum_{j=0}^{N-1}[(\lambda t)^j/j!]G^{(j)}(K)} \tag{A.3}$$

decreases strictly with t from $G(K)$ to $G^{(N+1)}(K)/G^{(N)}(K)$ when $N = 1, 2, \cdots$. In particular, when $N = 1$, $R_F(t, N) = 1 - G(K)$. When $N = 2, 3, \cdots$, differentiating $1 - R_F(t, N)$ with respect to t,

$$\frac{\lambda A_2(t, N)}{[\sum_{j=0}^{N-1}[(\lambda t)^j/j!]G^{(j)}(K)]^2},$$

where

$$A_2(t, N) \equiv \sum_{i=0}^{N-1}\frac{(\lambda t)^i}{i!}G^{(i)}(K)\sum_{j=0}^{N-2}\frac{(\lambda t)^j}{j!}G^{(j+2)}(K) - \sum_{i=0}^{N-1}\frac{(\lambda t)^i}{i!}G^{(i+1)}(K)\sum_{j=0}^{N-2}\frac{(\lambda t)^j}{j!}G^{(j+1)}(K). \tag{A.4}$$

By the similar method of Appendix 1,

$$A_2(t, N) = \sum_{i=0}^{N-1}\frac{(\lambda t)^i}{i!}\sum_{j=i}^{N-2}\frac{(\lambda t)^j}{(j+1)!}G^{(i)}(K)G^{(j+1)}(K)(G_{j+1} - G_i)(j - i + 1) < 0.$$

Furthermore,

$$\lim_{t\to 0}\frac{\sum_{j=0}^{N}[(\lambda t)^j/j!]G^{(j+1)}(K)}{\sum_{j=0}^{N}[(\lambda t)^j/j!]G^{(j)}(K)} = G(K),$$

and

$$\lim_{t\to\infty}\frac{\sum_{j=0}^{N}[(\lambda t)^j/j!]G^{(j+1)}(K)}{\sum_{j=0}^{N}[(\lambda t)^j/j!]G^{(j)}(K)} = \frac{G^{(N+1)}(K)}{G^{(N)}(K)}.$$

Thus, $1 - R_F(t, N)$ decreases with t from $G(K)$ to $G^{(N+1)}(K)/G^{(N)}(K)$, which completes the proof that $R_F(t, N)$ increases with t from $1-G(K)$ to $1-G^{(N+1)}(K)/G^{(N)}(K)$.

3. Prove that $R_L(t, N)$ increases with N. From (A.1),

$$\sum_{j=N}^{\infty}\frac{(\lambda t)^j}{j!}G^{(j+1)}(K)\sum_{j=N+1}^{\infty}\frac{(\lambda t)^j}{j!}G^{(j)}(K) - \sum_{j=N+1}^{\infty}\frac{(\lambda t)^j}{j!}G^{(j+1)}(K)\sum_{j=N}^{\infty}\frac{(\lambda t)^j}{j!}G^{(j)}(K)$$
$$= \frac{(\lambda t)^N}{N!}\sum_{j=N+1}^{\infty}\frac{(\lambda t)^j}{j!}G^{(j)}(K)G^{(N)}(K)\left[\frac{G^{(N+1)}(K)}{G^{(N)}(K)} - \frac{G^{(j+1)}(K)}{G^{(j)}(K)}\right] > 0. \tag{A.5}$$

For any $N_1 \geq N$,

$$\lim_{N\to\infty} \frac{\sum_{j=N}^{N_1}[(\lambda t)^j/j!]G^{(j+1)}(K)}{\sum_{j=N}^{N_1}[(\lambda t)^j/j!]G^{(j)}(K)} = \frac{G^{(N_1+1)}(K)}{G^{(N_1)}(K)}.$$

Because N_1 is arbitrary,

$$\lim_{N\to\infty} \frac{\sum_{j=N}^{\infty}[(\lambda t)^j/j!]G^{(j+1)}(K)}{\sum_{j=N}^{\infty}[(\lambda t)^j/j!]G^{(j)}(K)} = \lim_{N_1\to\infty} \frac{G^{(N_1+1)}(K)}{G^{(N_1)}(K)} = 0.$$

Thus, $1 - R_F(t, N)$ decreases with N from

$$\frac{\sum_{j=0}^{\infty}[(\lambda t)^j/j!]G^{(j+1)}(K)}{\sum_{j=0}^{\infty}[(\lambda t)^j/j!]G^{(j)}(K)} \tag{A.6}$$

to 0, which completes the proof that $R_L(t, N)$ increases with N.

4. Prove that $R_F(t, N)$ increases with N. From (A.3),

$$\begin{aligned}
&\sum_{j=0}^{N-1} \frac{(\lambda t)^j}{j!} G^{(j+1)}(K) \sum_{j=0}^{N} \frac{(\lambda t)^j}{j!} G^{(j)}(K) - \sum_{j=0}^{N} \frac{(\lambda t)^j}{j!} G^{(j+1)}(K) \sum_{j=0}^{N-1} \frac{(\lambda t)^j}{j!} G^{(j)}(K) \\
&= \frac{(\lambda t)^N}{N!} \sum_{j=0}^{N-1} \frac{(\lambda t)^j}{j!} G^{(j)}(K) G^{(N)}(K) \left[\frac{G^{(j+1)}(K)}{G^{(j)}(K)} - \frac{G^{(N+1)}(K)}{G^{(N)}(K)} \right] > 0. \qquad \text{(A.7)}
\end{aligned}$$

Thus, $1 - R_F(t, N)$ decreases with N from $G(K)$ to (A.6), which completes the proof that $R_F(t, N)$ increases with N.

Chapter 6

Garbage Collection Policies

It is an important problem to determine the tenuring collection time or major collection time to meet the pause time goal for a generational garbage collector. From such a viewpoint, this chapter proposes two stochastic models based on the working schemes of a generational garbage collector: Garbage collections occur at a nonhomogeneous Poisson process. Minor collections are made when the garbage collector begins to work, tenuring collection is made at a planned time T or at the first collection time when surviving objects have exceeded K for the first model. Major collection is made at time T or at the Nth collection for the second model. Using the techniques of cumulative processes in reliability theory, expected cost rates are obtained, and optimal policies of tenuring and major collection times which minimize them are discussed analytically and computed numerically.

6.1 Introduction

In recent years, generational garbage collection (Ungar, 1984; Vengerov, 2009) has been popular with programmers for the reason that it can be made more efficiently. Compared with classical tracing collectors, e.g., reference counting collector, mark-sweep collector, mark-compact collector and copying collector, a generational garbage collector is effective in computer programs with the character that it is unnecessary to mark or copy all active data of the whole heap for every collection, i.e, the collector concentrates effort on those objects which are most likely to be garbage. Based on the weak generational hypothesis (Ungar, 1984) which asserts that most objects are short-lived after their allocation, a generational garbage collector segregates objects by age into two or more regions called generations or multiple generations. The survival rates of younger generations are always much lower than those of older ones, which means that younger generations are more likely to be garbage and can be collected more frequently than older ones. Although such generational collections cost much shorter time than that of a full collection, the problems of pointers from older generations to younger ones and the size of root sets for younger generations will become more complicated. For these reasons, many generational collectors are limited to just two or three generations (Jones and Lins, 1996). This generational technique is now in widespread use for the memory management. For instance, the garbage collector, which is used in Sun's HotSpot Java Virtual Machine (JVM), manages heap space for both young and old generations (Vengerov, 2009): New objects space Eden, two equal survivor spaces SS♯1 and SS♯2 for surviving objects, and tenured objects space Old (Tenured), where Eden, SS♯1 and SS♯2 are for younger generations, and Old is for older ones.

The generational garbage collector uses minor collection and tenuring collection [1] for younger generations and major collection for multi-generations (Jones and Lins, 1996). Most generational garbage collectors are copying collectors, although it is possible to use mark-sweep collectors. In this chapter, we concentrate on a generational garbage collector using copying collection. However, for every garbage collection, the manner of stop and copy pauses all application threads to collect the garbage. The duration of time for which the collector has worked is called pause time (Jones and Lins, 1996), which is an important parameter for interactive systems, and depends largely upon the volume of surviving objects and the type of collections. That is, pause time suffered for minor collection increases with the number of collections and is less than that of tenuring collection, major collection

[1] Tenuring collection is also one kind of minor collections (Jones and Lins, 1996). We define tenuring collection in distinction from minor collection because there may be some surviving objects tenured from survivor space into Old.

pause time is the longest among the three ones.

This chapter considers a pause time goal which is called time cost or cost for simplicity, and our problem is to obtain optimal collection times which minimize the expected cost rates. Using the techniques of cumulative processes and reliability theory (Nakagawa, 2005; Nakagawa, 2007; Nakamura and Nakagawa, 2010), optimal tenuring collection times and major collection times are discussed analytically and computed numerically.

6.2 Working Schemes

In general, the frequency of garbage collections depends on whether the computer processes are busy or not. So that, it is practical to assume that garbage collections occur at a nonhomogeneous Poisson process with an intensity function $\lambda(t)$ and a mean-value function $R(t) \equiv \int_0^t \lambda(u)\mathrm{d}u$. Then, the probability that collections occur exactly j times in $(s,t]$ is

$$H_j(s,t) \equiv \frac{[R(t)-R(s)]^j}{j!}\mathrm{e}^{-[R(t)-R(s)]} \qquad (j=0,1,2,\cdots).$$

Denote $F_j(s,t)\,(j=1,2,\cdots)$ be the probability that collections occur at least j times in the time interval $(s,t]$,

$$F_j(s,t) = \int_s^t H_{j-1}(s,u)\lambda(u)\mathrm{d}u = \sum_{i=j}^{\infty} H_i(s,t), \tag{6.1}$$

where $F_0(s,t) \equiv 1$ and

$$H_j(t) \equiv H_j(0,t) = \frac{[R(t)]^j}{j!}\mathrm{e}^{-R(t)},$$
$$F_j(t) \equiv F_j(0,t) = \sum_{i=j}^{\infty} H_i(t).$$

The volume X_i of new objects in Eden at the ith collection has an identical distribution $G(x) \equiv \Pr\{X_i \le x\}$ $(i=1,2,\cdots)$. Further, compared with the method of estimating object lifetimes (Vengerov, 2009), it would be more easier to estimate the survivor rate of one group of objects using the statistical methods rather than focusing on every object. So that we suppose that survivor rate α_i ($0 \leq \alpha_i < 1; i = 1, 2, \cdots$), where $1 > \alpha_1 > \alpha_2 > \cdots > \alpha_i > \cdots \geq 0$, means that new objects will survive $100\alpha_i$ percent at the ith minor collection.

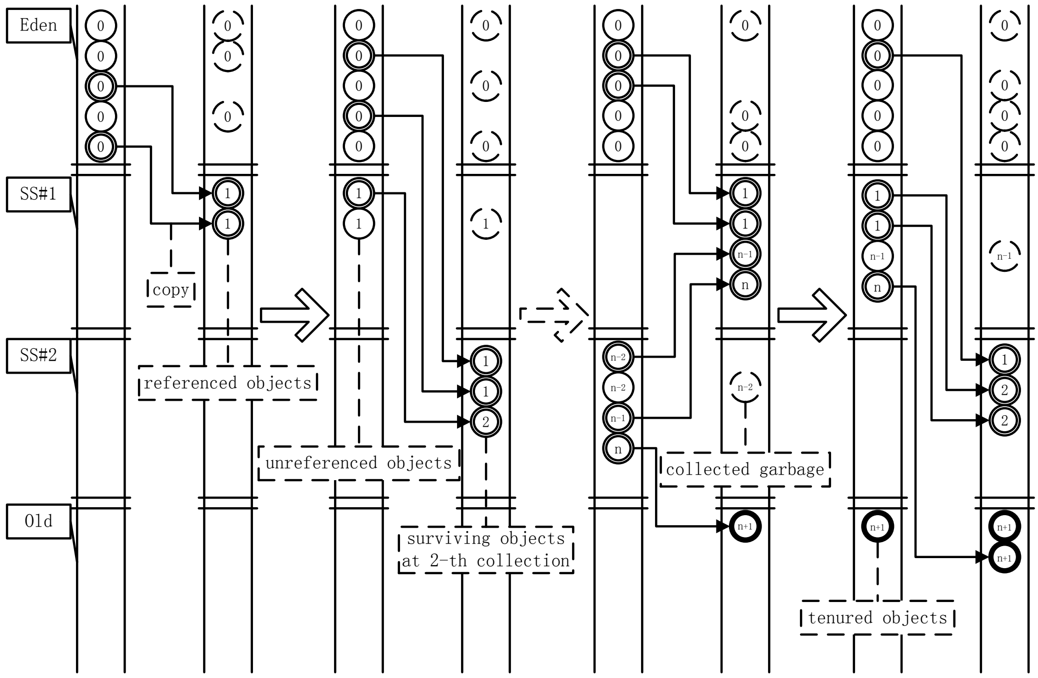

Figure 6.1: Working schemes of a generational garbage collector.

That is, detailed working schemes of a generational garbage collector which have been introduced in (Jones and Lins, 1996; Vengerov, 2009; Zhao, et al., 2010b, 2011b, 2012c) are given as following steps (Figure 6.1):

1. New objects X_1 are allocated in Eden.

2. When the first minor collection occurs, surviving objects $\alpha_1 X_1$ from Eden are copied into SS♯1.

3. When the second minor collection occurs, surviving objects $\alpha_1 X_2$ from Eden and $\alpha_2 X_1$ from SS♯1 are copied into SS♯2.

4. In the fashions of 1-3, minor collections copy surviving objects between SS♯1 and SS♯2 until they become tenured, i.e., tenuring collection occurs when

some parameter meets the tenuring threshold, and then, the older or the oldest objects are copied into Old.

5. When Old fills up, major collection of the whole heap occurs, and surviving objects from Old are kept in Old, while objects from Eden and survivor space are kept in survivor space.

In practice, tenuring threshold mentioned in step 4 above is adaptive, which is called adaptive tenuring (Jones and Lins, 1996) and can be modified at any time. In this chapter, we propose two cases of working schemes according to the properties of adaptive tenuring: Based on (Ungar, 1984), new objects can be tenured only if they survive at least one minor collection, because objects that survive two minor collections are much less than those that survive just one. In other words, surviving objects are likely to reduce slightly with the number of minor collections beyond the two. That is, for step 4:

4a. When tenuring collection occurs, surviving objects from Eden and survivor space are copied into the other survivor space and Old, respectively. That is, if tenuring collection is made at the jth ($j = 1, 2, \cdots$) collection, surviving objects $\alpha_1 X_j$ and $\alpha_2 X_{j-1} + \alpha_3 X_{j-2} + \cdots + \alpha_j X_1$ are copied into survivor space and Old, respectively.

4b. After tenuring collection, the same collection cycle begins with step 1. The collector works $1 \to 2 \to 3 \to$ 4a $\to$ 4b $\to 1 \to \cdots$. In this case, tenuring collections can be consider as renewal points of the collection processes, because Old will be filled with tenured objects slowly and major collection occurs rarely, especially when the tenuring threshold is high and the survivor rates are low. Modelings and optimizations of tenuring collection times are discussed in Sections 6.3.

From (Vengerov, 2009), the oldest objects can be tenured from survivor space into Old at every collection time when tenuring collection begins, i.e., for step 4:

4c. When tenuring collection occurs, the oldest objects from survivor space are copied into Old, and the other surviving objects from Eden and survivor space are copied into the other survivor space. That is, if tenuring collection is made at the jth ($j = 1, 2, \cdots$) collection, surviving objects $\alpha_1 X_j + \alpha_2 X_{j-1} + \cdots + \alpha_{j-1} X_2$ and $\alpha_j X_1$ are copied into survivor space and Old, respectively.

4d. When the next collection occurs, the collector works as the same rule as 4c. That is, when the second tenuring collection occurs, surviving objects $\alpha_1 X_{j+1} + \alpha_2 X_j + \cdots + \alpha_{j-1} X_3$ and $\alpha_j X_2$ are copied into the other survivor space and Old, respectively. The collector works $1 \to 2 \to 3 \to$ 4c $\to$ 4d $\to 5 \to 1 \to \cdots$. In this case, major collections can be consider as renewal points of the collection processes, because there are always some surviving objects tenured from survivor space into Old at every collection time when tenuring collection begins, especially when the tenuring threshold is low and the survivor rates are high. Related optimization problems of major collection times are discussed in Section 6.4.

From the above discussions, if tenuring collection is made at the jth ($j = 1, 2, \cdots$) collection, surviving objects that should be copied at the ith ($i = 0, 1, 2, \cdots, j-1$) minor collection, copied objects and tenured objects at the kth ($k = 1, 2, \cdots$) tenuring collection are, respectively,

$$\sum_{n=0}^{i-1} \alpha_{n+1} X_{i-n} < K, \qquad \sum_{n=1}^{j} \alpha_n X_{j+k-n} \geq K \qquad \text{and} \qquad \alpha_j X_k, \tag{6.2}$$

where $\sum_{n=0}^{-1} \equiv 0$, and K is tenuring threshold in step 4, which means that the total volume of surviving objects has exceeded level K. It could be easily seen that copied objects increase with the number of minor collections and are relatively stable with the number of tenuring collections. We define that the distribution of the total surviving objects at the ith minor collection is

$$G_i(x) \equiv \Pr\left\{\sum_{n=0}^{i-1} \alpha_{n+1} X_{i-n} \leq x\right\} \qquad (i = 0, 1, 2, \cdots), \tag{6.3}$$

where $G_i(x)$ deceases with i, and $G_0(x) \equiv 1$ means that there are no objects in the heap space at time 0. The probability that the total surviving objects exceed exactly a threshold level K at the $(i+1)$th ($i = 0, 1, 2, \cdots$) minor collection is

$$p_i(K) \equiv \int_0^K \overline{G}(K - x) \mathrm{d}G_i(x) = G_i(K) - G_{i+1}(K), \tag{6.4}$$

where $\overline{V}(x) \equiv 1 - V(x)$ for any distribution $V(x)$.

Let $c_S + c_M(x)$ be the cost suffered for every minor collection, where c_S is the constant cost of scanning surviving objects and x is the surviving objects that should be copied, $c_M(x)$ increases with x and $c_M(0) \equiv 0$. Then, the expected cost of the ith minor collection is

$$C(i, K) \equiv \frac{1}{G_i(K)} \int_0^K [c_S + c_M(x)] \,\mathrm{d}G_i(x) \qquad (i = 0, 1, 2, \cdots), \tag{6.5}$$

where $C(0, K) \equiv 0$ and $C(i, K)$ increases with i.

6.3 Tenuring Collection Time

6.3.1 Expected Cost Rate

Suppose that minor collections are made when the garbage collector begins to work, tenuring collection is made at a planned time T $(0 < T \leq \infty)$ or at the first collection time when surviving objects have exceeded a threshold level K $(0 < K \leq \infty)$, whichever occurs first. Then, the probability that tenuring collection is made at time T is

$$\mathrm{P}_T = \sum_{j=0}^{\infty} H_j(T) G_j(K), \tag{6.6}$$

and the probability that tenuring collection is made at level K is

$$\mathrm{P}_K = \sum_{j=0}^{\infty} F_{j+1}(T) p_j(K), \tag{6.7}$$

where note that $\mathrm{P}_T + \mathrm{P}_K \equiv 1$. The mean time to tenuring collection is

$$\begin{aligned} E_1(L) &= T \sum_{j=0}^{\infty} H_j(T) G_j(K) + \sum_{j=0}^{\infty} p_j(K) \int_0^T t \mathrm{d}F_{j+1}(t) \\ &= \sum_{j=0}^{\infty} G_j(K) \int_0^T H_j(t) \mathrm{d}t. \end{aligned} \tag{6.8}$$

The expected cost suffered for minor collections until tenuring collection is

$$C_M = \sum_{j=1}^{\infty} \sum_{i=1}^{j} C(i, K) H_j(T) G_j(K) + \sum_{j=1}^{\infty} \sum_{i=1}^{j} C(i, K) F_{j+1}(T) p_j(K)$$

$$= \sum_{j=1}^{\infty} C(j,K) F_j(T) G_j(K). \tag{6.9}$$

Then, the expected cost until tenuring collection is

$$E_1(C) = c_K - (c_K - c_T) \sum_{j=0}^{\infty} H_j(T) G_j(K) + \sum_{j=1}^{\infty} C(j,K) F_j(T) G_j(K), \tag{6.10}$$

where c_T and c_K (c_T, $c_K > c_S + c_M(K)$) are the costs suffered for tenuring collections at time T and when surviving objects have exceeded K, respectively. Therefore, from (6.8) and (6.10), by using the theory of renewal reward process (Ross, 1983), the expected cost rate is

$$C_1(T,K) = \frac{\begin{array}{c} c_K - (c_K - c_T) \sum_{j=0}^{\infty} H_j(T) G_j(K) \\ + \sum_{j=1}^{\infty} C(j,K) F_j(T) G_j(K) \end{array}}{\sum_{j=0}^{\infty} G_j(K) \int_0^T H_j(t) \mathrm{d}t}. \tag{6.11}$$

6.3.2 Optimal Policies

When tenuring collection is made only at time T,

$$C_1(T) \equiv \lim_{K \to \infty} C_1(T,K) = \frac{1}{T} \left\{ \sum_{j=1}^{\infty} F_j(T) \int_0^{\infty} [c_S + c_M(x)] \, \mathrm{d}G_j(x) + c_T \right\}. \tag{6.12}$$

Let $f_j(t)$ be a density function of $F_j(t)$, i.e., $f_j(t) \equiv \mathrm{d}F_j(t)/\mathrm{d}t$. Then, differentiating $C_1(T)$ with respect to T and setting it equal to zero,

$$\sum_{j=1}^{\infty} [T f_j(T) - F_j(T)] \int_0^{\infty} [c_S + c_M(x)] \, \mathrm{d}G_j(x) = c_T. \tag{6.13}$$

Let $L_1(T)$ be the left-hand side of (6.13),

$$L_1(0) \equiv \lim_{T \to 0} L(T) = 0,$$

$$L_1'(T) = \lambda'(T) T \sum_{j=0}^{\infty} H_j(T) \int_0^{\infty} [c_S + c_M(x)] \, \mathrm{d}G_{j+1}(x)$$

$$+ \lambda(T)^2 T \sum_{j=0}^{\infty} H_j(T) \int_0^{\infty} p_{j+1}(x) \mathrm{d}c_M(x).$$

Thus, if $\lambda(t)$ increases with t and $L_1(\infty) > c_T$, then there exists a finite and unique T_1^* $(0 < T_1^* < \infty)$ which satisfies (6.13), and the resulting cost rate is

$$C_1(T_1^*) = \lambda(T_1^*) \sum_{j=0}^{\infty} F_j(T_1^*) \int_0^{\infty} p_j(x) \mathrm{d}c_M(x).$$

In particular, when $H_j(t) = [(\lambda t)^j / j!]\mathrm{e}^{-\lambda t}$ $(j = 0, 1, 2, \cdots)$, i.e., garbage collections occur at a Poisson process with rate λ, (6.13) becomes

$$\sum_{j=1}^{\infty} j F_{j+1}(T) \int_0^{\infty} p_j(x) \mathrm{d}c_M(x) = c_T. \tag{6.14}$$

Differentiating the left-hand side of (6.14) with respect to T,

$$\lambda \sum_{j=1}^{\infty} j H_j(T) \int_0^{\infty} p_j(x) \mathrm{d}c_M(x) > 0.$$

Thus, if the left-hand side of (6.14) is greater than c_T, then there exists a finite and unique T_1^* $(0 < T_1^* < \infty)$ which satisfies (6.14).

When tenuring collection is made only at level K,

$$C_1(K) \equiv \lim_{T \to \infty} C_1(T, K) = \frac{\sum_{j=1}^{\infty} \int_0^K [c_S + c_M(x)] \mathrm{d}G_j(x) + c_K}{\sum_{j=0}^{\infty} G_j(K) \int_0^{\infty} H_j(t) \mathrm{d}t}. \tag{6.15}$$

Let $g_i(x)$ be a density function of $G_i(x)$ in (6.3), i.e., $g_i(x) \equiv \mathrm{d}G_i(x)/\mathrm{d}x$. Differentiating $C_1(K)$ with respect to K and setting it equal to zero,

$$Q_1(K) \sum_{j=0}^{\infty} G_j(K) \int_0^{\infty} H_j(t) \mathrm{d}t - \sum_{j=1}^{\infty} \int_0^K [c_S + c_M(x)] \, \mathrm{d}G_j(x) = c_K, \tag{6.16}$$

where

$$Q_1(K) \equiv \frac{[c_S + c_M(K)] \sum_{j=1}^{\infty} g_j(K)}{\sum_{j=1}^{\infty} g_j(K) \int_0^{\infty} H_j(t) \mathrm{d}t}.$$

Let $L_1(K)$ be the left-hand side of (6.16),

$$L_1(0) \equiv \lim_{K \to 0} L_1(K) = Q_1(0) \int_0^{\infty} H_0(t) \mathrm{d}t,$$

$$L_1'(K) = Q_1'(K) \sum_{j=0}^{\infty} G_j(K) \int_0^{\infty} H_j(t) \mathrm{d}t.$$

Thus, if $Q_1(K)$ increases with K and $L_1(0) < c_K < L_1(\infty)$, then there exists a finite and unique K_1^* $(0 < K_1^* < \infty)$ which satisfies (6.16), and the resulting cost rate is

$$C_1(K_1^*) = Q_1(K_1^*).$$

In particular, when $H_j(t) = [(\lambda t)^j/j!]\mathrm{e}^{-\lambda t}$, (6.16) becomes

$$c_M(K) + \int_0^K [c_M(K) - c_M(x)]\,\mathrm{d}M(x) = c_K - c_S, \tag{6.17}$$

whose left-hand side increases with K from 0 to ∞, where $M(x) \equiv \sum_{j=1}^{\infty} G_j(x)$. Thus, there exists a finite and unique K_1^* $(0 < K_1^* < \infty)$ which satisfies (6.17).

6.3.3 Numerical Examples

When $\lambda(t) = \lambda$, X_i $(i = 1, 2, \cdots)$ has a normal distribution $N(\mu, \sigma^2)$, $\alpha_i = \alpha/i$ $(0 \leq \alpha < 1; i = 1, 2, \cdots)$ and $c_M(x) = c_M x$. Then

$$F_j(t) = 1 - \sum_{i=0}^{j-1} \frac{(\lambda t)^i}{i!}\mathrm{e}^{-\lambda t}, \qquad G_j(x) = \Phi\left(\frac{x - \alpha\mu\nu_j}{\alpha\sigma\sqrt{\omega_j}}\right), \tag{6.18}$$

where $\Phi(x)$ is the standard normal distribution with mean 0 and variance 1, i.e., $\Phi(x) \equiv (1/\sqrt{2\pi}) \int_{-\infty}^{x} \mathrm{e}^{-u^2/2}\mathrm{d}u$, and

$$\nu_j \equiv \sum_{n=1}^{j} \frac{1}{n}, \qquad \omega_j \equiv \sum_{n=1}^{j} \frac{1}{n^2}.$$

Tables 6.1 and 6.2 present λT_1^*, $C_1(T_1^*)/\lambda$, K_1^* and $C_1(K_1^*)/\lambda$ for $c_T = c_K = 20$, 30, 40, $\mu = 8$, 10 and $\alpha = 0.40$, 0.45, 0.50, 0.55, 0.60 when $c_S = 10$, $c_M = 1$ and $\sigma = 1$. These show that optimal tenuring collection times λT_1^* increase with cost c_T and decrease with both the volume of new objects in Eden at collection time μ and the survivor rate α, optimal tenuring collection times K_1^* increase with all of c_K, μ and α, and $C_1(T_1^*)/\lambda$ and $C_1(K_1^*)/\lambda$ increase with all of c_T or c_K, μ and α.

We can explain all the results and obtain some interesting conclusions as follows:

Table 6.1: Optimal λT_1^* and $C_1(T_1^*)/\lambda$ when $c_S = 10$, $c_M = 1$ and $\sigma = 1$.

μ	α	$c_T = 20$		$c_T = 30$		$c_T = 40$	
		λT_1^*	$C_1(T_1^*)/\lambda$	λT_1^*	$C_1(T_1^*)/\lambda$	λT_1^*	$C_1(T_1^*)/\lambda$
8	0.40	8.99	19.24	12.48	20.18	15.84	20.89
	0.45	8.24	20.11	11.34	21.14	14.35	21.92
	0.50	7.61	20.95	10.42	22.07	13.15	22.92
	0.55	7.08	21.77	9.66	22.98	12.17	23.90
	0.60	6.64	22.58	9.03	23.86	11.34	24.85
10	0.40	7.61	20.95	10.42	22.07	13.14	22.91
	0.45	6.96	21.98	9.49	23.20	11.95	24.14
	0.50	6.44	22.97	8.75	24.30	10.97	25.31
	0.55	6.01	23.95	8.13	25.37	10.17	26.47
	0.60	5.64	24.91	7.61	26.43	9.49	27.60

1. When tenuring collection cost c_T or c_K increases, it is not economical to make tenuring collections frequently, then T_1^* or K_1^* should be postponed.

2. When μ or α increases, cost suffered for minor collections will increase in a shorter time, because of faster increase in copied objects. If cost c_T or c_K is constant in this case, T_1^* should be advanced. For K_1^*, it costs much shorter time to increase copied objects until level K, then K_1^* would increase suitably to decrease both the frequency of tenuring collections and the total minor collection cost.

3. The resulting cost rates $C_1(T_1^*)$ or $C_1(K_1^*)$ increase with all μ, α and c_T or c_K, because the total expected cost of one cycle increases but the expected time decreases.

4. It is interesting that $C_1(K_1^*)$ are always less than $C_1(T_1^*)$ for the same parameters, i.e., tenuring collections at level K are better than those at time T. In fact, from Tables 6.1 and 6.2, we can know that expected number of minor collections until tenuring collection for two models are almost the same. That

is, from the assumption of $\alpha_i = \alpha/i$, we can derive

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{[\lambda T_1^*]} < \frac{K_1^*}{\alpha\mu} < 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{[\lambda T_1^*]+1}, \tag{6.19}$$

where $[x]$ denotes the greatest integer contained in x. For example, when $c_T = c_K = 20$, $\mu = 8$ and $\alpha = 0.4$, $\lambda T_1^* = 8.99$ and $K_1^* = 8.76$, and hence

$$1+\frac{1}{2}+\cdots+\frac{1}{8} = 2.55 < \frac{8.76}{0.4\times 8} = 2.74 < 1+\frac{1}{2}+\cdots+\frac{1}{9} = 2.83.$$

We can estimate approximate values K_1^* from T_1^* using the relationship of the two policies in (6.19), and vice versa.

Table 6.2: Optimal K_1^* and $C_1(K_1^*)/\lambda$ when $c_S = 10$, $c_M = 1$ and $\sigma = 1$.

μ	α	$c_K = 20$		$c_K = 30$		$c_K = 40$	
		K_1^*	$C_1(K_1^*)/\lambda$	K_1^*	$C_1(K_1^*)/\lambda$	K_1^*	$C_1(K_1^*)/\lambda$
	0.40	8.76	18.76	9.61	19.61	10.71	20.71
	0.45	9.25	19.25	11.04	21.04	11.57	21.57
8	0.50	9.71	19.71	12.03	22.03	12.39	22.39
	0.55	10.12	20.12	12.69	22.69	13.18	23.18
	0.60	10.49	20.49	13.32	23.32	13.94	23.94
	0.40	10.72	20.72	12.05	22.05	12.41	22.41
	0.45	11.24	21.24	12.87	22.87	13.39	23.39
10	0.50	11.64	21.64	13.64	23.64	14.33	24.33
	0.55	12.08	22.08	14.37	24.37	15.23	25.23
	0.60	12.44	22.44	15.07	25.07	16.09	26.09

6.4 Major Collection Time

6.4.1 Including Minor and Tenuring Collections

6.4.1.1 Expected Cost Rate

Suppose that minor collections are made before surviving objects exceed a threshold level K $(0 < K < \infty)$, and when they have exceeded K, tenuring collections are always made. Further, major collection is made at time T $(0 < T \leq \infty)$ or at the Nth $(N = 1, 2, \cdots)$ collection including minor and tenuring collections (Figure 6.2), whichever occurs first.

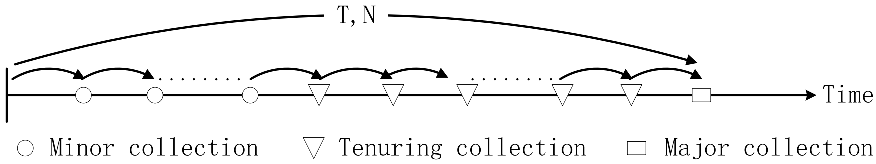

Figure 6.2: Major collection including minor and tenuring collections.

Let c_{kT} $(k = 1, 2, \cdots)$ be the cost suffered for the kth tenuring collection, where $c_S + c_M(K) < c_{1T} < c_{2T} < \cdots$, and c_F $(c_F > c_{kT})$ be the cost suffered for major collection. Then, the probability that major collection is made at time T is

$$\mathrm{P}_T = \sum_{j=0}^{N-1} H_j(T) G^{(j)}(K) + \sum_{j=1}^{N-1} \sum_{i=1}^{j-1} H_j(T) p_i(K) = 1 - F_N(T), \tag{6.20}$$

and the probability that major collection is made at collection N is

$$\mathrm{P}_N = F_N(T) G^{(N)}(K) + \sum_{j=0}^{N-1} F_N(T) p_j(K) = F_N(T), \tag{6.21}$$

where note that $\mathrm{P}_T + \mathrm{P}_N \equiv 1$. The mean time to major collection is

$$E_2(L) = \int_0^T t \mathrm{d}F_N(t) + T \sum_{j=0}^{N-1} H_j(T) = \int_0^T [1 - F_N(t)] \, \mathrm{d}t. \tag{6.22}$$

The expected costs suffered for minor collections and tenuring collections when major collection is made at time T are, respectively,

$$\begin{aligned} C_{TM} &= \sum_{j=1}^{N-1} H_j(T) \left[\sum_{i=1}^{j} C(i, K) G^{(j)}(K) + \sum_{i=1}^{j-1} \sum_{k=1}^{i} C(k, K) p_i(K) \right] \\ &= \sum_{j=1}^{N-1} H_j(T) \sum_{i=1}^{j} C(i, K) G^{(i)}(K), \end{aligned} \tag{6.23}$$

$$C_{TT} = \sum_{j=1}^{N-1} H_j(T) \sum_{i=0}^{j-1} \sum_{k=1}^{j-i} c_{kT} p_i(K)$$

$$= \sum_{j=1}^{N-1} H_j(T) \sum_{i=0}^{j-1} \left[c_{(i+1)T} - c_{(j-i)T} G^{(i+1)}(K) \right], \tag{6.24}$$

and the expected costs suffered for minor collections and tenuring collections when major collection is made at collection N are, respectively,

$$C_{NM} = F_N(T) \left[\sum_{j=1}^{N} C(j,K) G^{(N)}(K) + \sum_{j=1}^{N-1} \sum_{i=1}^{j} C(i,K) p_j(K) \right]$$
$$= F_N(T) \sum_{j=1}^{N} C(j,K) G^{(j)}(K), \tag{6.25}$$

$$C_{NT} = F_N(T) \sum_{j=0}^{N-1} \sum_{i=1}^{N-j} c_{iT} p_j(K)$$
$$= F_N(T) \sum_{j=0}^{N-1} \left[c_{(j+1)T} - c_{(N-j)T} G^{(j+1)}(K) \right]. \tag{6.26}$$

Thus, the total expected cost until major collection is, summing up from (6.23) to (6.26) and adding the cost c_F of major collection,

$$E_2(C) = c_F + \sum_{j=1}^{N} C(j,K) F_j(T) G^{(j)}(K)$$
$$+ \sum_{j=1}^{N} F_j(T) \left[c_{jT} - \sum_{i=0}^{j-1} G^{(j-i)}(K)(c_{(i+1)T} - c_{iT}) \right]. \tag{6.27}$$

Therefore, the expected cost rate is, from (6.22) and (6.27),

$$C_2(T,N) = \frac{c_F + \sum_{j=1}^{N} F_j(T) A_j}{\int_0^T [1 - F_N(t)] \mathrm{d}t}, \tag{6.28}$$

where

$$A_j \equiv c_{jT} + \int_0^K [c_S + c_M(x)] \, \mathrm{d}G^{(j)}(x) - \sum_{i=0}^{j-1} G^{(j-i)}(K)(c_{(i+1)T} - c_{iT}).$$

It can be easily proved that A_j increases with j because

$$A_{j+1} - A_j = (c_{1T} - c_S - c_M(K)) p_j(K) + \int_0^K p_j(x) \mathrm{d}c_M(x)$$
$$+ \sum_{i=1}^{j} p_{j-i}(K)(c_{(i+1)T} - c_{iT}) > 0.$$

6.4.1.2 Optimal Policies

When major collection is made only at time T,

$$C_2(T) \equiv \lim_{N\to\infty} C_2(T,N) = \frac{1}{T}\left[\sum_{j=1}^{\infty} F_j(T)A_j + c_F\right]. \tag{6.29}$$

Differentiating $C_2(T)$ in (6.29) with respect to T and setting it equal to zero,

$$\sum_{j=1}^{\infty} A_j \left[T\lambda(T)H_{j-1}(T) - F_j(T)\right] = c_F,$$

that is,

$$\sum_{j=1}^{\infty} A_j \int_0^T t\mathrm{d}\left[\lambda(t)H_{j-1}(t)\right] = c_F. \tag{6.30}$$

Let $L_2(T)$ be the left-hand side of (6.30),

$$L_2'(T) = \sum_{j=0}^{\infty} A_{j+1}\int_0^T t\lambda'(t)H_j(t)\mathrm{d}t + \sum_{j=0}^{\infty}(A_{j+2}-A_{j+1})\int_0^T t[\lambda(t)]^2 H_j(t)\mathrm{d}t,$$

$$L_2(\infty) = \sum_{j=1}^{\infty} A_j \int_0^{\infty} t\mathrm{d}[\lambda(t)H_{j-1}(t)].$$

Thus, if $\lambda(t)$ increases with t and $L_2(\infty) > c_F$, then there exists a finite and unique T_2^* $(0 < T_2^* < \infty)$ which satisfies (6.30).

In particular, when $\lambda(t) = \lambda$,

$$L_2'(T) = \sum_{j=0}^{\infty}(j+1)F_{j+2}(T)(A_{j+2}-A_{j+1}),$$

$$L_2(\infty) = \sum_{j=1}^{\infty}(A_\infty - A_j).$$

Therefore, if $\sum_{j=1}^{\infty}(A_\infty - A_j) > c_F$, then there exists a finite and unique T_2^* $(0 < T_2^* < \infty)$, and the resulting cost rate is

$$\frac{C_2(T_2^*)}{\lambda} = \sum_{j=0}^{\infty} H_j(T_2^*)A_{j+1}.$$

When major collection is made only at collection N,

$$C_2(N) \equiv \lim_{T\to\infty} C_2(T,N) = \frac{\sum_{j=1}^{N} A_j + c_F}{\int_0^\infty [1-F_N(t)]\mathrm{d}t} \qquad (N=1,2,\cdots). \tag{6.31}$$

From the inequality $C_2(N+1) - C_2(N) \geq 0$,

$$\sum_{j=0}^{N-1}\left[\frac{A_{N+1}}{\int_0^\infty H_N(t)\mathrm{d}t}\int_0^\infty H_j(t)\mathrm{d}t - A_{j+1}\right] \geq c_F. \tag{6.32}$$

Letting $L_2(N)$ be the left-hand side of (6.32),

$$L_2(N+1) - L_2(N) = \left[\frac{A_{N+2}}{\int_0^\infty H_{N+1}(t)\mathrm{d}t} - \frac{A_{N+1}}{\int_0^\infty H_N(t)\mathrm{d}t}\right]\int_0^\infty [1-F_{N+1}(t)]\,\mathrm{d}t. \tag{6.33}$$

Thus, if $A_{N+1}/\int_0^\infty H_N(t)\mathrm{d}t$ increases with N and $L_2(\infty) > c_F$, then there exists a finite and unique minimum N_2^* $(1 \leq N_2^* < \infty)$ which satisfies (6.32).

In particular, when $\lambda(t) = \lambda$,

$$L_2(N) = \sum_{j=1}^{N}(A_{N+1} - A_j),$$

$$L_2(N+1) - L_2(N) = (N+1)(A_{N+2} - A_{N+1}) > 0.$$

It is assumed that $A_\infty \equiv \lim_{j\to\infty} A_j < \infty$. Then,

$$L_2(\infty) = \sum_{j=1}^{\infty}(A_\infty - A_j).$$

Further, because $\sum_{j=1}^{N}(A_{N+1} - A_j) \geq A_{N+1} - A_1$ $(N=1,2,\cdots)$, if $A_\infty = \infty$, then $L_2(\infty) = \infty$. Therefore, if $\sum_{j=1}^{\infty}(A_\infty - A_j) > c_F$, then there exists a finite and unique minimum N_2^* $(1 \leq N_2^* < \infty)$, and the resulting cost rate is

$$A_{N_2^*} \leq \frac{C_2(N_2^*)}{\lambda} < A_{N_2^*+1}.$$

It is of interest that when collections occur at a Poisson process with rate λ, if $\sum_{j=1}^{\infty}(A_\infty - A_j) > c_F$, then both finite and unique T_2^* and N_2^* exist.

6.4.2 Including Tenuring Collections

6.4.2.1 Expected Cost Rate

Suppose that minor collections are made before surviving objects exceed a threshold level K, and after they have exceeded K, tenuring collections are always made. Further, major collection is made at time T $(0 < T \leq \infty)$ or at collection N $(N = 1, 2, \cdots)$ including tenuring collections (Figure 6.3), whichever occurs first.

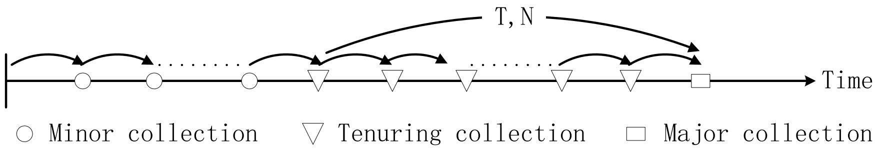

Figure 6.3: Major collection including tenuring collections.

Then, the probability that major collection is made at time T is

$$P_T = \sum_{j=0}^{\infty} \sum_{i=0}^{N-2} p_j(K) \int_0^{\infty} H_i(u, u+T) \mathrm{d}F_{j+1}(u), \tag{6.34}$$

and the probability that major collection is made at collection N is

$$P_N = \sum_{j=0}^{\infty} \sum_{i=N-1}^{\infty} p_j(K) \int_0^{\infty} H_i(u, u+T) \mathrm{d}F_{j+1}(u). \tag{6.35}$$

The mean time to major collection is

$$\begin{aligned}
E_3(L) = &\sum_{j=0}^{\infty} p_j(K) \int_0^{\infty} \left[\int_0^T (u+t) \mathrm{d}F_{N-1}(u, u+t) \right] \mathrm{d}F_{j+1}(u) \\
&+ \sum_{j=0}^{\infty} \sum_{i=0}^{N-2} p_j(K) \int_0^{\infty} (u+T) H_i(u, u+T) \mathrm{d}F_{j+1}(u) \\
= &\sum_{j=0}^{\infty} p_j(K) \int_0^{\infty} u \mathrm{d}F_{j+1}(u) \\
&+ \sum_{j=0}^{\infty} p_j(K) \int_0^{\infty} \left\{ \int_0^T [1 - F_{N-1}(u, u+t)] \, \mathrm{d}t \right\} \mathrm{d}F_{j+1}(u).
\end{aligned} \tag{6.36}$$

The expected costs suffered for minor collections and tenuring collections when major collection is made at time T are, respectively,

$$C_{TM} = \sum_{j=0}^{\infty}\sum_{i=1}^{j} c_{iM} p_j(K) \int_0^{\infty} [1 - F_{N-1}(u, u+T)]\, \mathrm{d}F_{j+1}(u), \tag{6.37}$$

$$C_{TT} = \sum_{j=0}^{\infty}\sum_{i=0}^{N-2}\sum_{k=1}^{i+1} c_{kT} p_j(K) \int_0^{\infty} H_i(u, u+T)\mathrm{d}F_{j+1}(u), \tag{6.38}$$

and the expected costs suffered for minor collections and tenuring collections when major collection is made at collection N are, respectively,

$$C_{NM} = \sum_{j=0}^{\infty}\sum_{i=1}^{j} c_{iM} p_j(K) \int_0^{\infty} F_{N-1}(u, u+T)\mathrm{d}F_{j+1}(u), \tag{6.39}$$

$$C_{NT} = \sum_{j=0}^{\infty}\sum_{i=1}^{N} c_{iT} p_j(K) \int_0^{\infty} F_{N-1}(u, u+T)\mathrm{d}F_{j+1}(u). \tag{6.40}$$

Thus, the total expected cost until major collection is, summing up from (6.37) to (6.40) and adding the cost c_F of major collection,

$$\begin{aligned} E_3(C) =& c_F + \sum_{j=1}^{\infty}\sum_{i=1}^{j} c_{iM} p_j(K) \\ &+ \sum_{j=0}^{\infty}\sum_{i=1}^{N} c_{iT} p_j(K) \int_0^{\infty} F_{i-1}(u, u+T)\mathrm{d}F_{j+1}(u). \end{aligned} \tag{6.41}$$

Therefore, from (6.36) and (6.41), the expected cost rate is

$$C_3(T, N) = E_3(C)/E_3(L). \tag{6.42}$$

6.4.2.2 Optimal Policies

When major collection is made only at time T,

$$C_3(T) \equiv \lim_{N\to\infty} C_3(T, N) = \frac{\begin{array}{l} c_F + \sum_{j=1}^{\infty}\sum_{i=1}^{j} c_{iM} p_j(K) \\ + \sum_{j=0}^{\infty}\sum_{i=1}^{\infty} c_{iT} p_j(K) \int_0^{\infty} F_{i-1}(u, u+T)\mathrm{d}F_{j+1}(u) \end{array}}{\sum_{j=0}^{\infty} p_j(K) \int_0^{\infty} u\mathrm{d}F_{j+1}(u) + T}. \tag{6.43}$$

Differentiating $C_3(T)$ with respect to T and setting it equal to zero,

$$\sum_{j=0}^{\infty} p_{j+1}(K) \int_0^{\infty} Q_3(u,T)\mathrm{d}F_{j+1}(u) = c_F + \sum_{j=1}^{\infty} c_{jM} G^{(j)}(K), \tag{6.44}$$

where

$$\begin{aligned} Q_3(u,T) \equiv & \sum_{i=1}^{\infty} c_{iT} \int_0^{\infty} (l+x)\, \mathrm{d}\,[\lambda(u+x) H_{i-2}(u,u+x)] \\ = & \sum_{i=1}^{\infty} c_{iT} \int_0^{\infty} (l+x)\, \lambda'(u+x) H_{i-2}(u,u+x)\mathrm{d}x \\ & + \sum_{i=1}^{\infty} (c_{(i+3)T} - c_{(i+2)T}) \int_0^{\infty} (l+x)\, [\lambda(u+x)]^2 H_i(u,u+x)\mathrm{d}x, \end{aligned}$$

and

$$l \equiv \sum_{j=1}^{\infty} p_j(K) \int_0^{\infty} t\mathrm{d}F_j(t),$$

which represents the mean time until surviving objects have exceeded K. Letting $L_3(T)$ be the left-hand side of (6.44). Thus, if $\lambda(t)$ increases with t, $L_3(T)$ increases with T. Therefore, if $L_3(\infty) > c_F + \sum_{j=1}^{\infty} c_{jM} G^{(j)}(K)$, then there exists a finite and unique T_3^* $(0 < T_3^* < \infty)$ which satisfies (6.44).

In particular, when $\lambda(t) = \lambda$, then $l = [1 + M(K)]/\lambda$, and

$$\begin{aligned} Q_3(u,T) = & [1 + M(K)] \sum_{j=1}^{\infty} F_j(T)(c_{(j+2)T} - c_{(j+1)T}) \\ & + \sum_{j=1}^{\infty} jF_{j+1}(T)(c_{(j+2)T} - c_{(j+1)T}), \\ L_3(\infty) = & \sum_{j=1}^{\infty} (c_{\infty T} - c_{(j+1)T}) + [1 + M(K)](c_{\infty T} - c_{2T}). \end{aligned}$$

Therefore, if

$$\sum_{j=1}^{\infty} (c_{\infty T} - c_{(j+1)T}) + [1 + M(K)](c_{\infty T} - c_{2T}) > c_F + \sum_{j=1}^{\infty} c_{jM} G^{(j)}(K),$$

there exists a finite and unique T_3^* $(0 < T_3^* < \infty)$, and the resulting cost rate is

$$\frac{C_3(T_3^*)}{\lambda} = \sum_{j=0}^{\infty} H_j(T_3^*) c_{(j+2)T}.$$

When major collection is made only at collection N,

$$C_3(N) \equiv \lim_{T\to\infty} C_3(T,N) = \frac{c_F + \sum_{j=1}^{\infty}\sum_{i=1}^{j} c_{iM} p_j(K) + \sum_{j=1}^{N} c_{jT}}{\sum_{j=0}^{\infty} p_j(K)\int_0^{\infty}[1-F_{j+N}(t)]\mathrm{d}t} \quad (N=1,2,\cdots). \tag{6.45}$$

From the inequality $C_3(N+1) - C_3(N) \geq 0$,

$$Q_3(N)c_{(N+1)T} - \sum_{j=1}^{N} c_{jT} \geq c_F + \sum_{j=1}^{\infty} c_{jM} G^{(j)}(K), \tag{6.46}$$

where

$$Q_3(N) \equiv \frac{\sum_{j=0}^{\infty} p_j(K)\int_0^{\infty}[1-F_{j+N}(t)]\mathrm{d}t}{\sum_{j=0}^{\infty} p_j(K)\int_0^{\infty} H_{j+N}(t)\mathrm{d}t}.$$

Letting $L_3(N)$ be the left-hand side of (6.46),

$$L_3(N+1) - L_3(N) = \left[\widetilde{Q}_3(N+1) - \widetilde{Q}_3(N)\right]\sum_{j=0}^{\infty} p_j(K)\int_0^{\infty}[1-F_{j+N+1}(t)]\,\mathrm{d}t,$$

where

$$\widetilde{Q}_3(i) \equiv \frac{c_{(i+1)T}}{\sum_{j=0}^{\infty} p_j(K)\int_0^{\infty} H_{j+i}(t)\mathrm{d}t}.$$

Thus, if $\widetilde{Q}_3(i)$ increases with i, $L_3(N)$ increases with N. Therefore, if $L_3(\infty) > c_F + \sum_{j=1}^{\infty} c_{jM}G^{(j)}(K)$, then there exists a finite and unique minimum N_3^* $(1 \leq N_3^* < \infty)$ which satisfies (6.46).

In particular, when $\lambda(t) = \lambda$, then $Q_3(N) = M(K) + N$, where $M(x) \equiv \sum_{j=1}^{\infty} G^{(j)}(x)$ is the expected number of minor collections before surviving objects exceed x, and

$$L_3(N) = \sum_{j=1}^{N}(c_{(N+1)T} - c_{jT}) + M(K)c_{(N+1)T},$$

$$L_3(N+1) - L_3(N) = [M(K) + N + 1]\left(c_{(N+2)T} - c_{(N+1)T}\right) > 0.$$

It is assumed that $c_{\infty T} \equiv \lim_{j\to\infty} c_{jT} < \infty$. Then,

$$L_3(\infty) = \sum_{j=1}^{\infty}(c_{\infty T} - c_{jT}) + M(K)c_{\infty T}.$$

Clearly, if $c_{\infty T} = \infty$, then $L_2(\infty) = \infty$. Therefore, if

$$\sum_{j=1}^{\infty}(c_{\infty T} - c_{jT}) + M(K)c_{\infty T} > c_F + \sum_{j=1}^{\infty} c_{jM}G^{(j)}(K),$$

then there exists a finite and unique minimum N_3^* $(1 \le N_3^* < \infty)$ which satisfies (6.46), and the resulting cost rate is

$$c_{N_3^*T} \le \frac{C_3(N_3^*)}{\lambda} < c_{(N_3^*+1)T}.$$

6.4.3 Numerical Examples

It is assumed that $c_{kT} = c_T + k\beta$ $(\beta > 0; k = 1, 2, \cdots)$, and other assumptions are the same as those in Section 6.3.3. Tables 6.3–6.6 present optimal λT_i^* and $C_i(T_i^*)/\lambda$ $(i = 2, 3)$, N_i^* and $C_i(N_i^*)/\lambda$ $(i = 2, 3)$, when $c_F = 100$, $c_T = c_N = 20$, $c_S = 10$, $c_M = 1$, $\mu = 10$ and $\sigma = 1$ for different α and β. These show that both λT_2^* and N_2^* decrease with α or β, both λT_3^* and N_3^* increase with α and decrease with β, all $C_i(T_i^*)/\lambda$ $(i = 2, 3)$ and $C_i(N_i^*)/\lambda$ $(i = 2, 3)$ increase with α or β.

Table 6.3: Optimal λT_2^* and $C_2(T_2^*)/\lambda$ for α and β.

α	$\beta = 1$		$\beta = 2$		$\beta = 5$	
	λT_2^*	$C_2(T_2^*)/\lambda$	λT_2^*	$C_2(T_2^*)/\lambda$	λT_2^*	$C_2(T_2^*)/\lambda$
0.3	17.98	24.4699	15.51	25.1134	13.14	26.0229
0.4	14.09	28.1939	10.66	31.5677	7.77	35.1867
0.5	13.80	31.6197	9.95	35.5256	6.60	42.2212
0.6	12.86	32.8499	9.86	37.6922	6.33	46.6393
0.7	12.86	33.6762	9.86	39.2143	6.27	50.0259

It can be explained as follows:

Table 6.4: Optimal N_2^* and $C_2(N_2^*)/\lambda$ for α and β.

α	$\beta=1$		$\beta=2$		$\beta=5$	
	N_2^*	$C_2(N_2^*)/\lambda$	N_2^*	$C_2(N_2^*)/\lambda$	N_2^*	$C_2(N_2^*)/\lambda$
0.3	17	24.1785	16	24.3642	15	24.7538
0.4	14	28.6941	11	30.5818	8	32.8485
0.5	14	31.1212	10	34.5257	7	39.7804
0.6	14	32.3506	10	36.6942	6	44.1853
0.7	14	33.1763	10	38.2160	6	47.5548

1. When α or β increases, it means that the total cost suffered for minor collections or tenuring collections increases, then optimal major collection times should be advanced, but even then the expected cost rates increase.

2. The differences between Tables 6.3 and 6.5, Tables 6.4 and 6.6, are that when α increases, $M(K)$ decreases, then optimal major collection times should be postponed, because it is not economic to make major collection frequently.

3. Compared Tables 6.3 with 6.4, Tables 6.5 with 6.6, these show that $C_2(T_2^*) > C_2(N_2^*)$ and $C_3(T_3^*) > C_3(N_3^*)$ for the same parameters, that is, major collections made at N_2 or N_3 are better than those at T_2 or T_3. It is interesting that $C_2(N_2^*) \approx C_3(N_3^*)$ and $C_2(T_2^*) \approx C_3(T_3^*)$, that is, although the two policies are different, the resulting expected cost rates are almost the same.

4. We can derive the relationship of the two polices, that is,

$$\lambda T_2^* \approx 1 + M(K) + \lambda T_3^*, \qquad N_2^* \approx M(K) + N_3^*.$$

   For example, when $\alpha = 0.3$ and $\beta = 1$, $M(K) = 14.8$, then

$$\lambda T_2^* = 17.98, \qquad 1 + M(K) + \lambda T_3^* = 1 + 14.8 + 1.95 = 17.75,$$
$$N_2^* = 17, \qquad M(K) + N_3^* = 14.8 + 3 = 17.8.$$

   Therefore, the concrete performances of the two kinds of policies would be depend on the program engineers and software system structures at the beginning, and so on.

Table 6.5: Optimal λT_3^* and $C_3(T_3^*)/\lambda$ for α and β.

α	$\beta = 1$		$\beta = 2$		$\beta = 5$	
	λT_3^*	$C_3(T_3^*)/\lambda$	λT_3^*	$C_3(T_3^*)/\lambda$	λT_3^*	$C_3(T_3^*)/\lambda$
0.3	1.95	23.9629	0.06	24.1471	0.01	24.3401
0.4	6.92	28.9179	3.42	30.8389	0.57	32.8532
0.5	9.45	31.4559	5.54	35.0679	2.10	40.4869
0.6	10.75	32.7357	6.69	37.3694	3.06	45.3683
0.7	11.60	33.5878	7.48	38.9639	3.80	49.0298

Table 6.6: Optimal N_3^* and $C_3(N_3^*)/\lambda$ for α and β.

	$\beta = 1$		$\beta = 2$		$\beta = 5$	
α	N_3^*	$C_3(N_3^*)/\lambda$	N_3^*	$C_3(N_3^*)/\lambda$	N_3^*	$C_3(N_3^*)/\lambda$
0.3	3	23.9071	2	24.1387	1	24.3365
0.4	8	28.6662	5	30.5026	2	32.5947
0.5	10	31.1223	7	34.4997	3	39.6811
0.6	12	32.3455	8	36.6799	4	44.1223
0.7	13	33.1731	9	38.2105	5	47.4478

It has been shown from tables 6.1–6.6 that the policies at level K and N are better than those at time T. However, from simple points, the policy at time T is easier to operate than those at level K and N. For further studies, we can modify the policy at time T from the viewpoint of operations. For example, it may be wasteful to collect the garbage for an operating system at a planned time T even if it is processing and would be better to do it after the process has been completed. In this case, we suppose that collection is always made at the first collection time after T, by using the overtime policy in Chapter 4. Then, $C_1(T)$ becomes

$$\widetilde{C}_1(T) = \frac{\sum_{j=1}^{\infty} F_j(T) \int_0^{\infty} [c_1 + c_0(x)] \mathrm{d}G_j(x) + c_2}{\sum_{j=0}^{\infty} \int_0^T \int_{T-u}^{\infty} (t+u) \mathrm{d}F_1(t) \mathrm{d}F_j(u)}, \tag{6.47}$$

and $C_3(T)$ becomes

$$\widetilde{C}_3(T) = \frac{\sum_{j=1}^{\infty} F_j(T)(c_3 - A_j) + c_4}{\sum_{j=0}^{\infty} \int_0^T \int_{T-u}^{\infty} (t+u)\mathrm{d}F_1(t)\mathrm{d}F_j(u)}. \tag{6.48}$$

It is feasible to discuss (6.47) and (6.48) analytically and compare them with $C_1(T)$ and $C_3(T)$ or with $C_2(K)$ and $C_4(N)$ numerically from the viewpoint of economy. To choose a better policy, it would depend on the original structure and actual operational scheme of a garbage collector.

6.5 Concluding Remarks

We have proposed the problems when to make tenuring collection and major collection to minimize the total expected pause time which can disturb the process of program in memory management. Two stochastic models based on the working schemes of a generational garbage collector have been discussed analytically: Garbage collections occur at a nonhomogeneous Poisson process. Minor collections are made when the garbage collector begins to work, tenuring collection is made at a planned time T or at the first collection time when surviving objects have exceeded K, major collection is made at time T or at the Nth collection. Using the techniques of cumulative processes and reliability theory, expected cost rates have been obtained, and optimal policies have been discussed analytically. Detailed examples and analysis have been given and compared numerically. Such theoretical methods could provide some useful information to computer programmers to design more efficient collectors in the near future.

Chapter 7

Conclusions

This dissertation has proposed several extended cumulative damage models and their applications to garbage collection policies in computer science, based on the standard cumulative processes (Cox, 1962; Nakagawa, 2007) in reliability theory. With respect to optimizations for the extended models, policies of a planned time T, shock, working, or periodic checking number N, and damage level Z have been optimized. For the application models of the generational garbage collector, practical working schemes with tenuring collection at time T or object level K and major collection at time T or collection N have been considered. The value contributions of the proposed models in this dissertation could be summarized as follows:

In Chapter 2, it may be more practical to assume that the initial damage level of a used system would be a random variable and the maintenances are imperfect. The models have indicated that failure rates with continuous and discrete times play an important role in deriving optimal policies and the necessity of optimizations also includes that cost for ICM should be greater than that for the first IPM which includes the maintenance cost for the initial damage. That is true because the bathtub curve in reliability shows that the used system has avoided the earlier failure problems period, and the higher damage level Y_0 is, the earlier maintenances should be made or the more wasteful operating a used system is.

In Chapter 3, if we take no account of failures caused by independent damage, the policies become the standard cumulative damage models, which is idealized in theory. However, in practice, take the fracture of brittle materials such as glasses for an example, maintenances for such a system should be more frequent because of minimal repairs. By combining additive and independent damages, three replacement policies with minimal repairs have been discussed from different viewpoints.

In Chapter 4, the damage has been defined as the results of works operated by the system when the every work time is a random variable. The proposed models have been formulated when every maintenance policy is made at the end of working time, which is more economical to operate. Especially when the damage models are applied to crack growth models for aircrafts, two failure facts have been considered, and the proposed models are more reasonable for such systems.

In Chapter 5, as an application of the notion “whichever occurs last”, the maintenance last damage model provides new analytical methods when two preventive maintenance policies are made, and the comparative studies between such a maintenance last and the conventional maintenance first have been shown that which policy is better depends on different cases. From (Zhao and Nakagawa, 2012), when a system is operating according to random working intervals, the policy with “whichever occurs last” can let the system operate for a longer time and avoid unnecessary maintenances. Such results are also suitable to the models when random working times proposed in chapter 4 are considered.

In Chapter 6, the pause time goal of a generational garbage collector has been selected to optimize tenuring and major collection times. The working schemes firstly proposed by (Vengerov, 2009), which were supposed by estimating the lifetimes of all objects, however, it would be more easier to estimate the survivor rate of one group of objects using the statistical methods proposed in this chapter.

We have obtained many results analytically and numerically. It has been shown in this dissertation that the optimal values are given by the unique and finite solution of equations under some reasonable conditions. To understand the results easily, we have given the numerical examples of each model, and have evaluated the results for several parameters. If some parameters are estimated from actual data, we can determine the best policy easily.

Bibliography

[Barlow and Proschan, 1965] Barlow, R.E., Proschan, F., 1965. Mathematical Theory of Reliability, Wiley, New York.

[Bogdanoff and Kozin, 1985] Bogdanoff, J.L., Kozin, F., 1985. Probabilistic Models of Cumulative Damage. John & Sons, New York.

[Brown and Proschan, 1983] Brown, M., Proschan, F., 1983. Imperfect repair. Journal of Applied Probability, 20, 851-859.

[Chan and Downs, 1978] Chan, P.K.W., Downs, T., 1978. Two criteria preventive maintenance. IEEE Transactions on Reliability, R-27, 272-273.

[Chen, et al., 2010a] Chen, M., Mizutani, S., Nakagawa, T., 2010. Random and age replacement policies. International Journal of Reliability, Quality and Safety Engineering, 17, 27-39.

[Chen, et al., 2010b] Chen, M., Nakamura, S., Nakagawa, T., 2010. Replacement and preventive maintenance models with random working times. IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences, E93-A, 500-507.

[Clinger and Rojas, 2006] Clinger, W.D., Rojas, F.V., 2006. Linear combinations of radioactive decay models for generational garbage collection. Science of Computer Programming, 62, 184-203.

[Cox, 1962] Cox, D.R., 1962. Renewal Theory, Metheun, London.

[Duchesne and Lawless, 2000] Duchesne, T., Lawless, J., 2000. Alternative time scales and failure time models. Lifetime Data Analysis, 6, 157-179.

[Esary, et al., 1973] Esary, J.D., Marshall, A.W., Proschan, F., 1973. Shock models and wear processes. Annals of Probability, 1, 627-649.

[Goda and Kitsuregawa, 2006] Goda, K., Kitsuregawa, M., 2006. Database partial reorganization for exploiting spacial locality of structural deterioration. DBSJ Letters, 1, 1-4.

[Gray and Reuter, 1992] Gray, J., and Reuter, A., 1992. Transaction Processing: Concepts and Techniques, Morgan Kaufmann, USA.

[Haerder and Reuter, 1983] Haerder, T., Reuter, A., 1983. Principles of transaction-oriented database recovery. ACM Computing Surveys, 15, 287-317.

[Hopp and Kuo, 1998] Hopp, W.J., Kuo, Y.L., 1998. An optimal structured policy for maintenance of partially observable aircraft engine components. Naval Research Logistics, 45, 335-352.

[Hudson, et al., 1997] Hudson, W.R., Haas, R., Uddin, W., 1997. Infrastructure Management, McGraw-Hill, New York.

[Jones and Lins, 1996] Jones, R., Lins, R., 1996. Garbage Collection: Algorithms for Automatic Dynamic Memory Management, John Wiley & Sons, Chichester.

[Kaldewaij and Vries, 2001] Kaldewaij, A., Vries, L., 2001. Optimal real-time garbage collection for acyclic pointer structures. Information Processing Letters, 77, 151-157.

[Kobbacy and Murthy, 2008] Kobbacy, K.A.H., Murthy, D.N.P., 2008. Complex System Maintenance Handbook, Springer, London.

[Lee and Chang, 2004] Lee, W.H., Chang, J.M., 2004. A garbage collection policy based on empirical behavior. Information Sciences, 167, 129-146.

[Lewis, et al., 2002] Lewis, M.L., Bernstein, B., Kifer, M., 2002. Databases and Transaction Processing: An Application-Oriented Approach, Addison Wesley, USA.

[Manzini, et al., 2010] Manzini, R., Regattieri, A., Pham, H., Ferrari, E., 2010. Maintenance for Industrial Systems, Springer, London.

[Murthy and Nguyen, 1981] Murthy, D.N.P., Nguyen, D.G., 1981. Optimal age-policy with imperfect preventive maintenance. IEEE Transactions on Reliability, R-30, 80-81.

[Muth, 1977] Muth, E.J., 1977. An optimal decision rule for repair vs replacement. IEEE Transactions on Reliability, R-26, 179-81.

[Nakagawa and Osaki, 1974] Some aspects of damage model. Microelectronics Reliability, 13: 253-257.

[Nakagawa, 1979] Nakagawa, T., 1979. Optimum replacement policies for a used unit. Journal of the Operations Research Society of Japan, 22, 338-47.

[Nakagawa, 2005] Nakagawa, T., 2005. Maintenance Theory of Reliability, Springer, London.

[Nakagawa, 2007] Nakagawa, T., 2007. Shock and Damage Models in Reliability Theory, Springer, London.

[Nakagawa, 2008] Nakagawa, T., 2008. Advanced Reliability Models and Maintenance Policies, Springer, London.

[Nakagawa and Ito, 2008] Nakagawa, T., Ito, K., 2008. Optimal maintenance policies for a system with multiechelon risks. IEEE Transactions on Systems, Man, and Cybernetics, 38, 461-469.

[Nakagawa, 2011] Nakagawa, T., 2011. Stochastic Processes with Applications to Reliability Theory, Springer, London.

[Nakagawa, et al., 2011] Nakagawa, T., Zhao, X., Yun, W., 2011. Optimal age replacement and inspection policies with random failure and replacement times. International Journal of Reliability, Quality and Safety Engineering, 18, 405-416.

[Nakagawa and Zhao, 2012] Nakagawa, T., Zhao, X., 2012. Optimization problems of a parallel system with a random number of units. IEEE Transactions on Reliability, 61, 543-548.

[Nakamura, et al., 2003] Nakamura, S., Qian, C., Fukumoto, S., et al., 2003. Optimal backup policy for a database system with incremental and full backup. Mathematical and computer modelling, 38,1373-1379.

[Nakamura and Nakagawa, 2010] Nakamura, S and Nakagawa, T (2010). Stochastic Reliability Modeling, Optimization and Applications, World Scientific, Singapore.

[Osaki, 2002] Osaki, S., 2002. Stochastic Models in Reliability and Maintenance, Springer, Berlin.

[Pinedo, 2002] Pinedo, M., 2002. Scheduling Theory, Algorithms, and Systems, Prentice-Hall, NJ.

[Qian, et al., 1999] Qian, C.H., Nakamura, S., Nakagawa, T., 1999. Cumulative damage model with two kinds of shocks and its application to the backup policy. Journal of the Operations Research Society of Japan, 42, 501-511.

[Qian, et al., 2002a] Qian, C.H., Pan, Y., Nakagawa, T., 2002. Optimal policies for a database system with two backup schemes. RAIRO Operations Research, 36, 227-235.

[Qian, et al., 2002b] Qian, C.H., Nakamura, S., Nakagawa, T., 2002. Optimal backup policies for a database system with incremental backup. Electronics and Communications in Japan, 85, 1-9.

[Qian, et al., 2005] Qian, C.H., Ito, K., Nakagawa, T., 2005. Optimal preventive maintenance policies for a shock model with given damage level. Journal of Quality in Maintenance Engineering, 11, 216-227.

[Qian, et al., 2010] Qian, C., Huang, Y., Zhao, X., Nakagawa, T., 2010. Optimal backup interval for a database system with full and periodic increment backup. Journal of Computers, 5, 557-564.

[Ross, 1983] Ross, S.M., 1983. Stochastic Process, Wiley, New York.

[Satow, et al., 1996a] Satow, T., Yasui, K., Nakagawa, T., 1996. Optimal garbage collection policies for a database in a computer system. RAIRO Operations Research, 30, 359-372.

[Satow, et al., 1996b] Satow, T., Yasui, K., Nakagawa, T., 1996. Optimal garbage collection policies for a database with random threshold level. Electronics and Communications in Japan, 79, 31-40.

[Scarf, et al., 1996] Scarf, P.A., Wang, W., Laycock, P.J., 1996. A stochastic model of crack growth under periodic inspections. Reliability Engineering & System Safety, 51, 331-339.

[Schijve, 1995] Schijve, J., 1995. Multiple-site damage in aircraft fuselage structure. Fatigue & Fracture of Engineering Materials & Structures, 18, 329-344.

[Sheu and Griffith, 1996] Sheu, S.H., Griffith, W.S., 1996. Optimal number of minimal repairs before replacement of a system subject to shocks. Naval Research Logistics, 43, 319-333.

[Sheu, 1998] Sheu, S.H., 1998. A generalised age and block replacement of a system subject to shocks. European Journal of Operational Research, 108, 345-362.

[Sheu and Griffith, 2002] Sheu, S.H., Griffith, W.S., 2002. Extend block replacement policy with shock models and used items. European Journal of Operational Research, 140, 50-60.

[Sheu and Chien, 2004] Sheu, S.H., Chien, Y.H., 2004. Optimal age-replacement policy of a system subject to shocks with random lead-time. European Journal of Operational Research, 159, 132-144.

[Sheu et al., 2012] Sheu S.H., Chang, C.C., Zhang, Z.G., Chien, Y.H., 2012. A note on replacement policy for a system subject to non-homogeneous pure birth shocks. European Journal of Operational Research, 216, 503-508.

[Sobczyk and Trebicki, 1989] Sobczyk, K., Trebicki, J., 1989. Modelling of random fatigue by cumulative jump processes. Engineering Fracture Mechanics, 34, 477-493.

[Soman and Krintz, 2007] Soman, S., Krintz, C., 2007. Application-specific garbage collection. Journal of Systems and Software, 80, 1037-1056.

[Stadje, 2003] Stadje, W., 2003. Renewal analysis of a replacement process. Operations Research Letters, 31, 1-6.

[Sugiura, et al., 2004] Sugiura, T., Mizutani, S., Nakagawa, T., 2004. Optimal random replacement policies. In: Pham, H., Yamada, S., (Eds), Proceedings of the Tenth ISSAT International Conference on Reliability and Quality in Design, Las Vegas, USA, 99-103.

[Ungar, 1984] Ungar, D., 1984. Generation scavenging: A non-disruptive high performance storage reclamation algorithm. ACM Sigplan Notices, 19, 157-167.

[Ungar and Jackson, 1992] Ungar, D., Jackson, F., 1992. An adaptive tenuring policy for generation scavengers. ACM Transactions on Programming Languages and Systems, 14, 1-27.

[Vengerov, 2009] Vengerov, D., 2009. Modeling, analysis and throughput optimization of a generational garbage collector. Technical Report, Sun Labs, TR-2009-179.

[Wang and Pham, 2003] Wang, H., Pham, H., 2003. Optimal imperfect maintenance models. In: Pham, H., (ed), Handbook of Reliability Engineering, Springer, London, 397-414.

[Wang and Pham, 2007] Wang, H., Pham, H., 2007. Reliability and Optimal Maintenance, Springer, London.

[William and Tuel, 1978] William, G., Tuel, J.R., 1978. Optimum reorganization points for linearly growing files. ACM Transactions on Database Systems, 1, 32-40.

[Wortman et al., 1994] Wortman, M.A., Klutke, G.A., and Ayhan, H., 1994. A maintenance strategy for systems subjected to deterioration governed by random shocks. IEEE Transactions on Reliability, 43, 439-445.

[Zhao, et al., 2009] Zhao, X., Qian, C., Nakagawa T., 2009. Study on preventive software rejuvenation policy for two kinds of bugs. Journal of Systems Science and Information, 7, 103-110.

[Zhao, et al., 2010a] Zhao, X., Nakagawa, T., 2010. Optimal replacement policies for damage models with the limit number of shocks. International Journal of Reliability and Quality Performance, 2, 13-20.

[Zhao, et al., 2010b] Zhao, X., Nakamura, S., Nakagawa, T., 2010. Optimal policies for random and periodic garbage collections with tenuring threshold. In: Tomar, G., et al., (Eds), Communications in Computer and Information Science 74, Springer, 125-135.

[Zhao, et al., 2011a] Zhao, X., Nakayama, K., Nakamura, S., 2011. Cumulative damage models with replacement last. In: Kim, T., et al., (Eds), Communications in Computer and Information Science 257, Springer, 338-345.

[Zhao, et al., 2011b] Zhao, X., Nakamura, S., Nakagawa, T., 2011. Two Generational garbage collection models with major collection time. IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences, E94-A, 1558-1566.

[Zhao, et al., 2012a] Zhao, X., Zhang, H., Qian, C., Nakagawa, T., Nakamura, S., 2012. Replacement models for combining additive independent damages. International Journal of Performability Engineering, 8, 91-100.

[Zhao, et al., 2012b] Zhao, X., Nakagawa, T.,Qian, C., 2012. Optimal imperfect preventive maintenance policies for a used system. International Journal of Systems Science, 43, 1632-1641.

[Zhao, et al., 2012c] Zhao, X., Nakamura, S., Nakagawa, T., 2012. Optimal tenuring and major collection times for a generational garbage collector. Asia-Pacific Journal of Operational Research, DOI: 10.1142/S0217595912400180.

[Zhao and Nakagawa, 2012] Zhao, X., Nakagawa, T., 2012. Optimization problems of replacement first or last in reliability theory. European Journal of Operational Research, 223, 141-149.

Publication

Chapter 2

1. X. Zhao, T. Nakagawa, C. Qian, 2012. **Optimal imperfect preventive maintenance policies for a used system**. International Journal of Systems Science, 43, 1632-1641.

Chapter 3

1. X. Zhao, H. Zhang, C. Qian, T. Nakagawa, S. Nakamura, 2012. **Replacement models for combining additive independent damages**. International Journal of Performability Engineering, 8, 91-100.

2. X. Zhao, C. Qian, S. Nakamura, 2011. **Optimal replacement policies for three continuous damage models**. In: L. Cui and X. Zhao (Eds), Proceedings of the 7th International Conference on Mathematical Methods in Reliability: Theory, Methods, Applications, Beijing, China, pp.188-194.

3. X. Zhao, C. Qian, S. Nakamura, 2011. **Optimal replacement policy for continuous damage model and its application to garbage collection**. In: H. Huang, et al. (Eds), International Conference on Quality, Reliability, Risk, Maintenance, and Safety Engineering, Xi'an, China, pp.400-4204.

4. X. Zhao, M. Chen, T. Nakagawa, 2010. **Three kinds of replacement models combined with additive and independent damages**. In: X. Zhang,

et al. (Eds), The Ninth International Symposium on Operations Research and Its Applications, Chengdu, China, pp.31-38.

Chapter 4

1. X. Zhao, S. Nakamura, T. Nakagawa, 2013. **Optimal maintenance policies for cumulative damage models with random working times**. Journal of Quality in Maintenance Engineering, to appear in Vol.19 in 2013.

2. X. Zhao, T. Nakagawa, 2010. **Optimal replacement policies for damage models with the limit number of shocks**. International Journal of Reliability and Quality Performance, 2, 13-20.

3. X. Zhao, C. Qian, T. Nakagawa, 2012. **Age replacement with overtime policy**. In: H. Yamamoto, et al. (Eds), The 5th Asia-Pacific International Symposium on Advanced Reliability and Maintenance Modeling V, Nanjing, China, pp.661-668.

4. X. Zhao, T. Nakagawa, 2011. **Optimal age replacement and inspection policies with random failure and replacement times**. In: H. Pham (Ed), Proceedings of the 17th ISSAT International Conference on Reliability and Quality in Design, Vancouver, B.C., Canada, pp.200-204.

5. X. Zhao, T. Nakagawa, K. Iwata, 2010. **Optimal maintenance policies for cumulative damage models with random working times**. In: S. Chukova, et al. (Eds), Proceedings 4th Asia-Pacific International Symposium, Advanced Reliability Modeling IV, Wellington, New Zealand, pp.835-842.

Chapter 5

1. X. Zhao, T. Nakagawa, 2013. **Comparisons of periodic and random replacement policies**. In: I. Frenkel, et al. (Eds), Modern Reliability Theory and Risk Analysis. Wiley, accepted in 2012 and to appear in 2013.

2. X. Zhao, C. Qian, T. Nakagawa, 2013. **Optimal policies for cumulative damage models with maintenance last and first**. Reliability Engineering & System Safety, 110, 50-59.

3. X. Zhao, T. Nakagawa, 2012. **Optimization problems of replacement first or last in reliability theory**. European Journal of Operational Research, 223, 141-149.

4. X. Zhao, T. Nakagawa, 2012. **A summary of newly proposed age replacement policies**. In: H. Pham (Ed), Proceedings of the 18th ISSAT International Conference on Reliability and Quality in Design, Boston, USA, 2012, pp.106-110.

5. X. Zhao, K. Nakayama, S. Nakamura, 2011. **Cumulative damage models with replacement last**. In: T. Kim, et al. (Eds), Communications in Computer and Information Science 257, Springer, pp.338-345.

Chapter 6

1. X. Zhao, S. Nakamura, C. Qian, 2013. **Generational garbage collection policies**. In: T. Dohi and T. Nakagawa (Eds), Stochastic Reliability and Maintenance Modeling, Springer, accepted in 2012 and to appear in 2013.

2. X. Zhao, S. Nakamura, T. Nakagawa, 2012. **Optimal tenuring and major collection times for a generational garbage collector**. Asia-Pacific Journal of Operational Research, 29, DOI: 10.1142/S0217595912400180.

3. X. Zhao, S. Nakamura, T. Nakagawa, 2011. **Two Generational garbage collection models with major collection time**. IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences, E94-A, 1558-1566.

4. X. Zhao, C. Qian, S. Nakamura, 2011. **Studies on tenuring collection times for a generational garbage collector**. In: X. Zhang, et al. (Eds),

The Tenth International Symposium on Operations Research and Its Applications, Dunhuang, China, pp.284-291.

5. X. Zhao, S. Nakamura, T. Nakagawa, 2010. **Optimal policies for random and periodic garbage collections with tenuring threshold**. In: G. Tomar, et al. (Eds), Communications in Computer and Information Science 74, Springer, pp.125-135.

6. X. Zhao, T. Nakagawa, S. Nakamura, 2010. **Studies on optimal collection time for a generational garbage collector**. In: S. Chukova, et al. (Eds), Proceedings 4th Asia-Pacific International Symposium, Advanced Reliability Modeling IV, Wellington, New Zealand, pp.843-850.

7. X. Zhao, S. Nakamura, C. Qian, 2010. **Optimal policies for a generational garbage collector with tenuring threshold**. In: X. Zhang, et al. (Eds), The Ninth International Symposium on Operations Research and Its Applications, Chengdu, China, pp.39-45.