Studies on
Optimal Maintenance Policies
for
Extended Inspection Models

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February 2004

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by
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Dissertation submitted in partial fulfillment
for the degree of Doctor of Engineering

Under the supervision of
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Aichi Institute of Technology
Toyota, Japan

February 2004

Abstract

The thesis studies optimal inspection and maintenance policies for high reliable systems. Some modified and extended inspection models from the viewpoint of actual models are considered. Using the reliability theory, such models are mathematically analyzed and useful inspection schedules are determined. Reasonable costs of inspections and failures for each model are introduced, and the expected costs until the detection of failures are obtained. Optimal inspection policies which minimize these expected costs are derived analytically and numerically. In particular, these results would be practically applied to determine inspection schedules for systems such as digital control devices. Further, optimal maintenance and inspection policies for a finite interval are similarly considered and are analytically discussed.

This thesis is divided into 7 chapters. An initial chapter gives the introduction which is constructed by the review of literatures and the organization of this thesis. Chapters 2 to 4 consider the modified inspection models and discuss these optimal policies: Chapter 2 studies optimal inspection policies for a two-unit system. First, the system operates as a two-unit system, and when one unit fails, it operates as a single-unit system. The system is checked continuously or periodically while it operates as a two-unit system, and is checked periodically by self-diagnosis after a failed unit is detached from the system. Chapter 3 studies optimal inspection policies for a system with self-testing which can detect some failures without performing external

inspection. However, the failure might not be detected rapidly by self-testing, and so, it would be necessary to check the system periodically by inspection. This chapter considers the model where a failure is detected by either self-testing or periodic inspection. Then, optimal inspection policies which minimize the expected costs are analytically derived. Chapter 4 studies optimal maintenance and inspection policies for a finite interval. Optimal policies which minimize the expected costs of periodic replacement with minimal repair, block replacement, simple replacement and inspection policy are derived for a finite interval. Chapter 5 studies optimal inspection policies for a system with two types of inspection: There might exist some failures which can not be detected by type-1 inspection and can be detected only by type-2 inspection, however, type-1 inspection has a lower cost than that of type-2 inspection. An optimal number to perform type-1 inspection until the next type-2 inspection is analytically derived. Chapter 6 considers an extended model in Chapter 5, where the system is replaced at the specified N-th type-2 inspection. The expected cost per unit of time is analytically obtained, and an optimal number to perform type-1 inspection until the next type-2 inspection is numerically derived.

Finally, in Chapter 7, the results are summarized.

Acknowledgment

The author would be like to appreciate Professor Toshio Nakagawa, the supervisor of my study for his constant guidance, encouragement and suggestions throughout this work.

The author wishes to thank the members of this thesis reviewing committee: Professor Kazumi Yasui, Professor Naohiro Ishi, Professor Tomio Kurokawa, and Professor Kazuyuki Teramoto for their careful reviews of this dissertation.

The author is also grateful to Professor Shunji Osaki of Nanzan University for having presented the papers at some national conferences, and wishes to thank Kodo Ito of Mitsubishi Heavy Industries, LTD., Processor Syouji Nakamura of Kinjo Gakuin University, Professor Hiroaki Sandoh of University of Marketing & Distibution Sciences and all members of Nagoya Computer and Reliability Research Group for their useful comments and discussions.

Furthermore, the author would like to thank all professors of Aichi Institute of Technology for continual support for this study.

This dissertation could not have been accomplished without the guidance and encouragement of the above members.

Finally, the author wishes to thank my family for their mental and various supports.

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Chapter 1

Introduction

In recent years, many systems such as digital control devices and other devices for information processing have been greatly developed and become widely used. Therefore, the improvement of their reliability has become necessary and important. For instance, some failures of systems might incur great losses, and sometimes, might cause a social confusion. The complexity of systems has dramatically increased, and as a result, it has become much more difficult to predict the occurrence of failures. Therefore, it is indispensably necessary and greatly important to check systems suitably and detect their failures by inspection. However, the cost of inspection would be usually very expensive and it would be difficult to develop the method of inspection to detect any failure. Therefore, it is of great interest to determine appropriate schedules of inspection from the viewpoints of reliability and economics.

For the purpose to assure of the reliability and economics, the numerical evaluation of activities for inspection and maintenance have been great importance with globalization and deregulation of the world. Many reliability researchers have studied theoretical and practical problems to evaluate and improve the reliability and economics for complex phenomena of real systems, using mainly stochastic processes. The reliability theory has been actually applied to evaluate these criterion in several practical fields such as industrial, mechanical and electronic engineerings. Further, this theory has been also applied to information, network and communication systems.

In this thesis, we form some stochastic models in which systems are checked to detect their failures by inspection at suitable times by inspection. We are mainly interested in optimal scheduled times of inspection which minimize the total expected cost from the beginning of system operation to the detection of failure and the expected cost per unit of time as objective functions. We are also concerned with the expected costs and optimal policies when systems have to operate for a finite interval. Further, we give numerical examples at each chapter to understand the results easily and make some useful discussions for them.

1.1 Inspection Policy

Some failures of systems might incur great losses, and sometimes, might cause a social confusion. Hence, it is necessary to check systems at suitable times and to detect early their failures. However, it might incur much loss cost and work when inspection is done so frequently. Therefore, by making a trade-off between the loss cost of failure and the cost of inspection, we have to determine optimal schedules of inspection.

Optimal inspection policies have been established as a great part of reliability theory. In this thesis, we consider one cycle from the beginning of system operation to the detection of failure, and adopt the expected cost on one cycle as an objective function. Then, we discuss optimal inspection policies which minimize the total expected cost of one cycle and the expected cost per unit of time, which is given by [Ross (1970)]

$$\frac{\text{Expected cost per cycle}}{\text{Expected time per cycle}}.$$

Barlow and Proschan (1965) summarized the inspections policies which minimize two expected costs until the detection of failure and per unit of time. Ross (1970) and

Osaki (1992) explained plainly the stochastic processes and applied them to typical stochastic and reliability models. Ben-Daya and Duffuaa (2000), and Gertsbakh (2000) overviewed many maintenance policies. However, all failures can not be detected upon inspection. The imperfect-inspection model was first treated in Weiss (1962), Coleman and Abrams (1962), and Morey (1967). Apostolakis and Bansal (1977) considered imperfect inspections due to human errors, and Srivastava and Wu (1993) estimated the parameter of an exponential failure distribution, using the maximum likelihood method. Osaki (2002) and Pham (2003) edited the reliability books with advanced researches and applications, and summarized extensively optimal maintenance policies.

Most faults occur intermittently in digital systems. Su *et al.* (1978), Koren and Su (1979), Nakagawa *et al.* (1989, 1990) discussed optimal periodic tests to detect intermittent faults. Chung (1995) developed a simple algorithm to compute an optimal time, and Ismaeel and Bhatnagar (1997) introduced a random test for detection of faults in combinational circuits.

Inspection models have been recently applied to many actual systems: Christer *et al.* (1982, 1984, 1989) reported the inspection maintenances of building, industrial plant and underwater structure. Sim *et al.* (1984a, 1984b, 1985) analyzed the periodic test of combustion turbine units and standby equipments in dormant systems and nuclear generating stations. Further, the following inspections were made: Fail-safe structures by Young (1984), manufacturing stations by Cassandras and Han (1992), automatic trips and warning instruments by Sherwin (1995), bearings by Garners *et al.* (1998). Ito and Nakagawa (2000, 2004) discussed optimal policies for FADEC (Full-Authority Digital Engine Control) which is a control device of gas turbine engines and mainly consists of a two-unit system.

At present, some terrible industrial accidents of plants have happened in Japan, and caused serious damage and great losses to a modern society. Risk management for such plants has become more and more important, however, theoretical arguments for these problems have not been advanced sufficiently. The results and techniques showed in inspection policies would be useful and helpful for maintaining in good condition of many real systems.

1.2 Technique for Detection of Failure

It is great important to develop advanced techniques for the detection of failure. However, the development and realization become increasingly difficult because systems have become larger and more complex than ever before. Actually, several useful methods to detect failures have been proposed: Jha and Gupta (2003) summarized the techniques to test digital systems in detail. O'Connor (2001) surveyed widely the technologies related with tests for electronic circuits. Lala (2001) summarized fault-tolerant design techniques with self-checking of digital circuits.

A simple inspection method for systems such as digital circuits is the comparison-checking with outputs of two-unit system. The performance of the system might not degrade, however, it would be often expensive to configure it. In this case, it is also necessary to determine mechanically which units has failed and is detached from the system [Nanya (1991)].

One method to check analogue circuits is mainly measurements of parameters such as voltage, resistance, impedance, and so on. The basic approach to test digital circuits is to check whether output codes for an assumed output set are correct or not. To detect failures certainly, some codes which are called *test pattern* should be inputed in systems and output codes are checked directly. However, as the complexity of systems have greatly increased, it has become very difficult to design the test pattern to detect any failure, and moreover, the time to perform the test has become too long. Other popular and simple methods are watch-dog timers and watch-dog processors, which interrupt some signals and check the responses periodically [Touma (1990), Nanya (1991)].

In general, the properties of self-checking, which involves those of fault-secure and self-testing, are required to design high reliable systems. Fault-secure means the property that a failed system outputs either the correct code or codes which are not in an assumed output code space. That is, the system with fault-secure does not output incorrect codes which are not required as the result of input codes. Self-testing means that a failed system outputs codes which are not in an assumed code space for at least one input code, that is, the system with self-testing can detect any failure without performing external inspection. However, the realization of perfect design with self-checking for complex systems would be impossible [Lala (2001), O'Connor (2001)].

In this thesis, we treat some inspection policies for several systems such as digital control devices for aircraft engines. The inspections with above methods for such systems are realized as periodic external inspections with tester or self-diagnosis, and it is necessary to determine suitable schedule times of inspection.

1.3 Outline of Thesis

This section describes the outline of this thesis. This thesis is divided into Introduction, Chapter 2-6, Conclusions and Bibliography.

Chapter 2 considers inspection policies for a two-unit system. The system firstly operates as a two-unit system and is checked by comparison-checking. When one unit fails, the system operates as a single-unit system and is checked periodically by self-diagnosis. We introduce two costs of one check for a two-unit system and for a single-unit system which are not the same with each other. In the model, two cases of continuous and periodic comparison-checking are adopted, and the self-diagnosis checks the system periodically for each case. The total expected cost and expected cost per unit of time are derived, and optimal inspection intervals which minimize the expected cost per unit of time are analytically discussed. Numerical examples are presented when the failure time of each unit has an exponential distribution.

Chapter 3 considers inspection policies for a system with self-testing. The system with self-testing can detect its failure during its operating state without external inspection. However, the detection by self-testing might has the latency, *i.e.*, some failures might not be detected rapidly. Therefore, to achieve a high reliability, the system should be checked by the external inspection at scheduled times. Thus, if the system fails, then its failure is detected by self-testing or at the next periodic inspection, whichever occur first. The total expected cost and expected cost per unit of time are derived, and optimal inspection intervals which minimize them are analytically discussed. Numerical examples are presented when the failure time has an exponential distribution.

Chapter 4 considers maintenance and inspection policies when a system has to operate for a finite interval. In actual fields, most systems have a finite span of their use. Using the partition method for this problem, a finite interval is divided into equal parts of maintenance or inspection [Nakagawa (2004)]. Optimal policies which minimize the expected costs of periodic replacement with minimal repair, block replacement, simple replacement and inspection policy are derived for a finite interval. Further, we show how to compute optimal checking times numerically when the failure time has a Weibull distribution and gamma distribution.

Chapter 5 considers inspection policies for a system which is checked by two types of inspection: Type-1 inspection has a lower cost than that of type-2 inspection, and type-1 inspection checks the system more frequently than type-2 inspection. However, there exist some failures which can not be detected by type-1 inspection and can be detected only by type-2 inspection. It is assumed that failures are classified into two cases where they can be detected by type-1 inspection and not detected with certain probability. We derive analytically an optimal number to perform type-1 inspection until the next type-2 inspection. Finally, numerical examples are given when the failure time distribution is exponential.

Chapter 6 considers a replacement policy for the same inspection model as in Chapter 5: The system with two types of inspection is replaced at the specified number of type-2 inspection. We derive an optimal number to perform type-1 inspection until the next type-2 inspection. Numerical examples are computed when the failure time distribution is exponential.

Finally, Chapter 7 summarizes the result derived in this thesis.

Chapter 2

Optimal Inspection Policies with Comparison-Checking for a Two-Unit System

This chapter considers optimal inspection policies for a two-unit system. First, the system is checked continuously or periodically by comparison-checking. When one unit fails, the failed unit is detected by comparison-checking and the system operates as a single-unit system. After that, the system is checked periodically by self-diagnosis. The total expected cost and expected cost per unit of time are derived, and optimal inspection policies which minimize the expected cost per unit of time are analytically discussed. Numerical examples are given when the failure time has an exponential distribution.

2.1 Introduction

In this chapter, we consider optimal inspection policies for a two-unit system such as digital control devices for aircraft engines. It is assumed that the system has input and output codes sequentially. When the system starts to operate, both units are in operational state, and their outputs are compared with each other to detect its failure early. That is, the system is checked by comparison-checking, which is commonly used because its implemental method is relatively easy. When the failure is detected by comparison-checking, the failed unit is detached and the system operates as a single-unit system. After that, the system is checked periodically by self-diagnosis. As actual examples of self-diagnosis, there are watch-dog timer, inputing test pattern codes, methods to check the parameters such as voltage, resistance, impedance, and so on [O'Connor (2001)].

We consider two models of comparison-checking model:
(1) *Continuous comparison-checking model*: When the system operates as a two-unit system, it is checked continuously by comparison-checking. In other words, the comparison-checking is done for each output code sequentially. Thus, incorrect output codes of a failed unit can be detected immediately. But, the loss cost for performing the comparison-checking increases, because a larger number of comparison-checking is executed. Thus, this model should be used in the case where high reliability and quality outputs are acquired.
(2) *Periodic comparison-checking model*: One unit (unit A) is connected with the part of output, and the other unit (unit B) operates as a standby unit (see Figure 2.1). The system is checked periodically by comparison-checking. It is assumed for simplicity that the intervals of comparison-checking for a two-unit system and of periodic self-diagnosis for a single-unit system are the same. In this model, although the latency time to detect failures of a two-unit system may occur, the system will be able to have a sufficient reliability.

A typical example for such systems is FADEC (Full-Authority Digital Engine Control): FADEC is a digital control unit of gas-turbine engines for systems such as aircrafts, and in general, consists of as a two-unit system to require a hight reliability. Recently, FADEC's are used as control units for general industrial gas-turbine engines, and

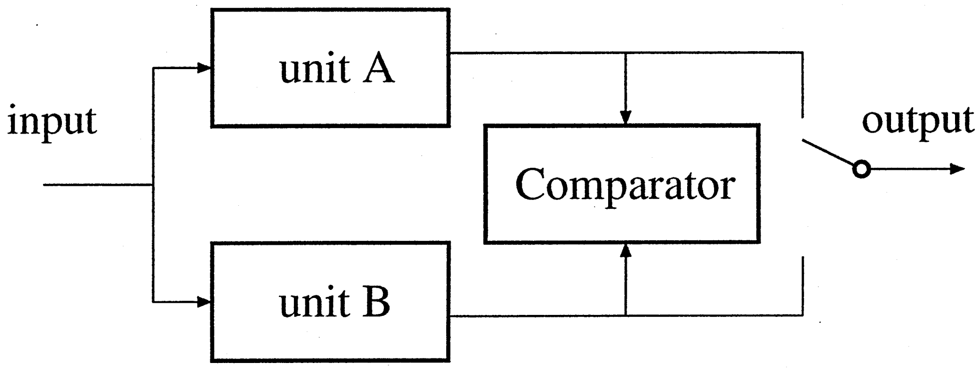

Figure 2.1: System with two units.

several researchers evaluated the reliability and interval of their self-diagnosis [Kodo Ito and Nakagawa (2000, 2003), Hjelmgren *et al.* (1998), Elks *et al.* (2000)].

It is supposed that when the failure of a single-unit system is detected and removed, the system becomes like new and starts to operate again as a two-unit system. Furthermore, the inspections by comparison-checking and self-diagnosis can detect any failure. We obtain the expected costs analytically, and derive the optimal inspection intervals which minimize them. Numerical examples are finally given when the failure time has an exponential distribution.

2.2 Model and Assumptions

2.2.1 Continuous comparison-checking model

Consider a system which is configured as a two-unit system whose outputs are checked by comparison-checking for each processing time. When failures of the system occur, they can be detected immediately by comparison-checking. But, more frequent number of comparison-checking increases the loss cost for degradation of system performance, so that, it should be used only in the case where high reliability and quality output are acquired.

For the above model, we define the following assumptions:

(i) When the first unit fails at time t_1, the system is switched to a single-unit system, and the next failure of the other unit occurs at time t_2.

(ii) While the system operates as a two-unit system, it is checked continuously by comparison-checking and can detect any failure instantly, *i.e.*, any failure is detected at time t_1. When the failure is detected, the failed unit is detached and the system operates as a single-unit system. After that, a single-unit system is checked by self-diagnosis at periodic times $t_1 + kT$ $(k = 1, 2, \dots)$ (see Figure 2.2). Any replacement or maintenance before the detection of failure is not considered.

(iii) The failure time distribution of each unit has an independent and identical general distribution $F(t)$ with finite mean $1/\lambda$, where $\overline{F}(t) \equiv 1 - F(t)$.

(iv) A cost c_d is the loss cost per unit of time for the time elapsed between a failure of a single-unit system and its detection at the next time of inspection, and c_r is the constant cost for maintenance or replacement, when the second failure is detected by self-diagnosis.

(v) A cost c_{e1} is the loss cost per unit of time for the time elapsed between the beginning of system operation and the detection of failure at time t_1 by comparison-checking, and c_i is the cost for one check by self-diagnosis, where $c_d/\lambda > c_r$ and $c_d > c_{e1}$.

(vi) After detecting the first failure of a two-unit system, the loss cost and time, and the performance degradation needed to locate and detach the failed unit are negligible.

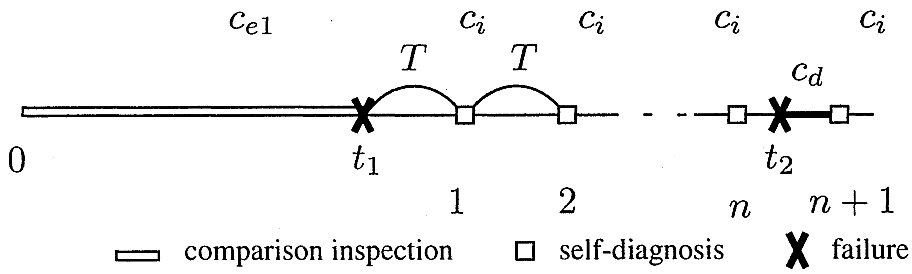

Figure 2.2: Comparison-checking model with each processing time.

We define one cycle as the time from the beginning of system operation to the detection of the second failure. Then, the mean time of one cycle is

$$2\int_0^\infty\Big\{\sum_{n=0}^\infty\int_{nT+t_1}^{(n+1)T+t_1}[(n+1)T+t_1]dF(t_2)\Big\}dF(t_1)$$

$$=2T\int_0^\infty\sum_{n=0}^\infty\overline{F}(nT+t_1)dF(t_1)+\int_0^\infty\overline{F}(t_1)^2dt_1. \tag{2.1}$$

Further, the total expected cost of one cycle is

$$2\int_0^\infty\Big[\sum_{n=0}^\infty\int_{nT+t_1}^{(n+1)T+t_1}\big\{c_{e1}t_1+c_i(n+1)+c_d\big[(n+1)T+t_1-t_2\big]\big\}dF(t_2)\Big]dF(t_1)+c_r$$

$$=2\int_0^\infty\Big\{(c_i+c_dT)\sum_{n=0}^\infty(n+1)\Big[\overline{F}(nT+t_1)-\overline{F}((n+1)T+t_1)\Big]$$

$$+\int_{t_1}^\infty\big[(c_{e1}+c_d)t_1-c_dt_2\big]dF(t_2)\Big\}dF(t_1)+c_r$$

$$=2\int_0^\infty\Big[(c_i+c_dT)\sum_{n=0}^\infty\overline{F}(nT+t_1)-c_d\int_{t_1}^\infty\overline{F}(t_2)dt_2+c_{e1}t_1\overline{F}(t_1)\Big]dF(t_1)+c_r. \tag{2.2}$$

The reason of two times on the first terms in (2.1) and (2.2) is that either failure of unit 1 or unit 2 may occur.

Thus, the expected cost $C_c(T)$ per unit of time is

$$C_c(T) = \frac{c_r + 2\int_0^\infty \left[(c_i + c_d T)\sum_{n=0}^\infty \overline{F}(nT + t_1) - c_d\int_{t_1}^\infty \overline{F}(t_2)dt_2 + c_{e1}t_1\overline{F}(t_1)\right]dF(t_1)}{2T\int_0^\infty \sum_{n=0}^\infty \overline{F}(nT + t_1)dF(t_1) + \int_0^\infty \overline{F}(t_1)^2 dt_1}. \tag{2.3}$$

Obviously,

$$C_c(0) \equiv \lim_{T\to 0} C_c(T) = \infty, \tag{2.4}$$

$$C_c(\infty) \equiv \lim_{T\to\infty} C_c(T) = c_d. \tag{2.5}$$

Therefore, there exists an optimal interval T_c^* $(0 < T_c^* \leq \infty)$ of inspection which minimizes $C_c(T)$.

2.2.2 Periodic comparison-checking model

Considers the system which is checked periodically by comparison-checking at the same interval as that of self-diagnosis. The difference from the previous model is the interval of comparison-checking for a two-unit system. In this model, the reliability might be less than the previous model because the latency to detect failures of a two-unit system occurs, but, the performance degradation due to comparison-checking might be smaller. Generally, if this system has sufficient reliability then it would be more realistic than the previous model.

For this model, we define the following assumptions:

(i) Both intervals of comparison-checking for a two-unit system and self-diagnosis for a single-unit system are the same. That is, the system is checked always at periodic times kT $(k = 1, 2, \dots)$, irrespective of the number of operating units.

(ii) A cost c_{e2} is the loss cost per unit of time for one check by comparison-checking for a two-unit system, and c_i is the cost for one check by self-diagnosis for a single-unit system.

(iii) Unit A which is first connected to output fails at time t_a $(0 < t_a < \infty)$ and unit B which is first in standby fails at time t_b $(0 < t_b < \infty)$.

(iv) Make the same assumptions as (iii), (iv) and (vi) in the previous model.

The total expected cost of one cycle is classified into the following three cases:

1) $kT < t_a \leq (k+1)T \leq mT < t_b \leq (m+1)T$

Suppose that unit A fails during $(kT, (k+1)T]$ $(k = 0, 1, \cdots)$ before unit B fails, and after that, unit B fails during $(mT, (m+1)T]$ $(m = k+1, k+2, \cdots)$ (see Figure 2.3). Then, the expected cost of one cycle is

$$\sum_{m=1}^{\infty} \int_{mT}^{(m+1)T} \Big[\sum_{k=0}^{m-1} \int_{kT}^{(k+1)T} \{ c_{e2}(k+1) + c_i(m-k) + c_d\big[(k+1)T - t_a + (m+1)T - t_b\big] \} dF(t_a) \Big] dF(t_b). \tag{2.6}$$

2) $kT < t_b \leq (k+1)T \leq mT < t_a \leq (m+1)T$

Suppose that unit B fails during $(kT, (k+1)T]$ $(k = 0, 1, \cdots)$ before unit A fails, and it is detached from the system. Thereafter, unit A fails during $(mT, (m+1)T]$ $(m = k+1, k+2, \cdots)$ (see Figure 2.4). Then, the expected cost of one cycle is

$$\sum_{m=1}^{\infty} \int_{mT}^{(m+1)T} \Big\{ \sum_{k=0}^{m-1} \int_{kT}^{(k+1)T} \big[c_{e2}(k+1) + c_i(m-k) + c_d\big((m+1)T - t_a\big) \big] dF(t_b) \Big\} dF(t_a). \tag{2.7}$$

3) $kT < t_a, t_b \leq (k+1)T$

Suppose that both units A and B fail during $(kT, (k+1)T]$ $(k = 0, 1, 2, \cdots)$, and their failures are detected by the next comparison-checking (see Figure 2.5). Then, the

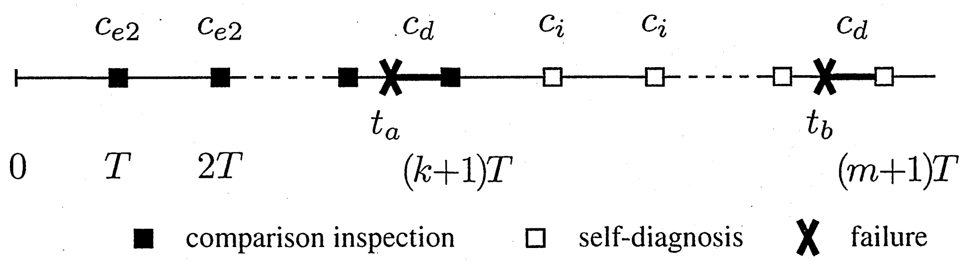

Figure 2.3: Case 1 of periodic comparison-checking.

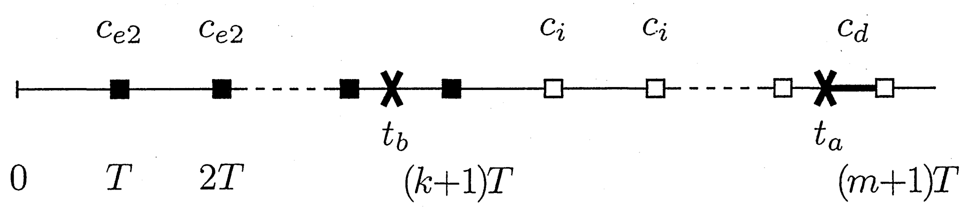

Figure 2.4: Case 2 of periodic comparison-checking.

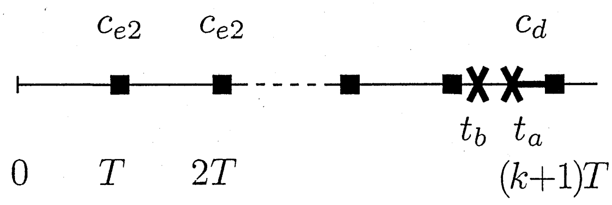

Figure 2.5: Case 3 of periodic comparison-checking.

expected cost of one cycle is

$$\sum_{k=0}^{\infty}\int_{kT}^{(k+1)T}\big[c_{e2}(k+1)+c_d\big((k+1)T-t_a\big)\big]\big[F((k+1)T)-F(kT)\big]dF(t_a). \tag{2.8}$$

Thus, the total expected cost of one cycle is obtained by summing equations (2.6), (2.7) and (2.8), and by adding the maintenance cost c_r as follows (see Appendix 2.1) :

$$\begin{aligned} c_r + (c_{e2}-c_i)\sum_{m=0}^{\infty}\overline{F}(mT)^2 + c_i\sum_{m=0}^{\infty}\big[1-F(mT)^2\big] \\ + c_d\sum_{m=0}^{\infty}\big[1+F(mT)\big]\int_{mT}^{(m+1)T}\big[F(t)-F(mT)\big]dt. \end{aligned} \tag{2.9}$$

Similarly, the mean time of one cycle is

$$\begin{aligned} &2\sum_{m=1}^{\infty}\int_{mT}^{(m+1)T}\Big[\sum_{k=0}^{m-1}\int_{kT}^{(k+1)T}(m+1)TdF(t_b)\Big]dF(t_a) \\ &+\sum_{k=0}^{\infty}\int_{kT}^{(k+1)T}(k+1)T\big[F((k+1)T)-F(kT)\big]dF(t_a) \\ &= T\Big[2\sum_{m=0}^{\infty}\overline{F}(mT)-\sum_{m=0}^{\infty}\overline{F}(mT)^2\Big] \\ &= T\sum_{m=0}^{\infty}\big[1-F(mT)^2\big]. \end{aligned} \tag{2.10}$$

Thus, the expected cost $C_p(T)$ per unit of time is, from (2.9) and (2.10),

$$C_p(T) = \frac{\left[\begin{array}{l} c_r + (c_{e2}-c_i)\sum_{m=0}^{\infty}\overline{F}(mT)^2 + c_i\sum_{m=0}^{\infty}\big[1-F(mT)^2\big] \\ + c_d\sum_{m=0}^{\infty}\big[1+F(mT)\big]\int_{mT}^{(m+1)T}\big[F(t)-F(mT)\big]dt \end{array}\right]}{T\sum_{m=0}^{\infty}\big[1-F(mT)^2\big]}. \tag{2.11}$$

Obviously,

$$C_p(0) \equiv \lim_{T\to 0} C_p(T) = \infty, \tag{2.12}$$

$$C_p(\infty) \equiv \lim_{T\to\infty} C_p(T) = c_d. \tag{2.13}$$

Therefore, there exists an optimal self-diagnosis interval T_p^* $(0 < T_p^* \leq \infty)$ of inspection which minimizes $C_p(T)$.

2.3 Optimal Inspection Policy

2.3.1 Continuous comparison-checking model

Consider the continuous comparison-checking when the failure time of each unit has an exponential distribution $F(t) = 1 - e^{-\lambda t}$. Then, the total expected cost of one cycle in (2.3) is rewritten as

$$2\int_0^\infty \left[(c_i + c_d T)\sum_{n=0}^{\infty} e^{-\lambda(nT+t_1)} - c_d\int_{t_1}^{\infty} e^{-\lambda t_2}dt_2 + c_{e1}t_1 e^{-\lambda t_1}\right]\lambda e^{-\lambda t_1}dt_1 + c_r$$

$$= c_i\left(\frac{1}{1-e^{-\lambda T}}\right) + c_d\left(\frac{T}{1-e^{-\lambda T}} - \frac{1}{\lambda}\right) + \frac{c_{e1}}{2\lambda} + c_r. \tag{2.14}$$

The mean time of one cycle in (2.1) is

$$T\int_0^\infty \sum_{n=0}^{\infty} 2\lambda e^{-\lambda(nT+2t_1)}dt_1 + \int_0^\infty e^{-2\lambda t_1}dt_1 = \frac{T}{1-e^{-\lambda T}} + \frac{1}{2\lambda}. \tag{2.15}$$

Thus, the expected cost $C_c(T)$ per unit of time is

$$C_c(T) = c_d + \frac{2\lambda c_i - (3c_d - 2\lambda c_r - c_{e1})(1-e^{-\lambda T})}{2\lambda T + 1 - e^{-\lambda T}}. \tag{2.16}$$

We find an optimal self-diagnosis interval T_c^* which minimizes the expected cost $C_c(T)$. Differentiating $C_c(T)$ with respect to T and putting it equal to 0, we have

$$(3c_d - 2\lambda c_r - c_{e1})\frac{1-(1+\lambda T)e^{-\lambda T}}{\lambda} + c_i(1-e^{-\lambda T}) = 3c_i. \tag{2.17}$$

Letting denote the left-hand side of (2.17) by $Q_c(T)$,

$$Q_c(0) \equiv \lim_{T \to 0} Q_c(T) = 0, \tag{2.18}$$

$$Q_c(\infty) \equiv \lim_{T \to \infty} Q_c(T) = \frac{3c_d - 2\lambda c_r - c_{e1}}{\lambda} + c_i, \tag{2.19}$$

$$Q_c'(T) = \lambda e^{-\lambda T}\big[(3c_d - 2\lambda c_r - c_{e1})T + c_i\big]. \tag{2.20}$$

It can be easily proved that $Q_c(T)$ is strictly increasing from 0 to $Q_c(\infty)$ from the assumptions of $c_d/\lambda > c_r$ and $c_d > c_{e1}$.

Therefore, we have the following optimal policy:

(i) If $(3c_d - 2\lambda c_r - c_{e1}) > 2\lambda c_i$ then there exists a finite and unique T_c^* $(0 < T_c^* < \infty)$ which satisfies (2.17).

(ii) If $(3c_d - 2\lambda c_r - c_{e1}) \le 2\lambda c_i$ then $T_c^* = \infty$, *i.e.*, no periodic inspection should be made and $C_c(\infty) = c_d$.

In general, since $c_d/\lambda > c_r + c_i$, case (ii) will not occur in practice.

2.3.2 Periodic comparison-checking model

Consider the periodic comparison-checking when the failure time of each unit has an exponential distribution $F(t) = 1 - e^{-\lambda t}$. Then, the total expected cost in (2.9) is rewritten as

$$\begin{aligned} & c_r + (c_{e2} - c_i)\sum_{m=0}^{\infty} e^{-2\lambda m T} + c_i \sum_{m=0}^{\infty}\left(2e^{-\lambda m T} - e^{-2\lambda m T}\right) \\ & + c_d \sum_{m=0}^{\infty}\left(2 - e^{-\lambda m T}\right)\int_{mT}^{(m+1)T}\left(e^{-\lambda m T} - e^{-\lambda t}\right) dt \\ = {} & c_r + \frac{c_{e2} + 2c_i e^{-\lambda T} + c_d[T(1 + 2e^{-\lambda T}) - (1 + e^{-\lambda T} - 2e^{-2\lambda T})/\lambda]}{1 - e^{-2\lambda T}}. \end{aligned} \tag{2.21}$$

The mean time in (2.10) is rewritten as

$$T\sum_{m=0}^{\infty}\left(2e^{-\lambda mT}-e^{-2\lambda mT}\right)=T\frac{1+2e^{-\lambda T}}{1-e^{-2\lambda T}}. \tag{2.22}$$

Thus, the expected cost $C_p(T)$ per unit of time is

$$C_p(T)=c_d+\frac{c_{e2}+2c_ie^{-\lambda T}+c_r(1-e^{-2\lambda T})-c_d(1+e^{-\lambda T}-2e^{-2\lambda T})/\lambda}{T(1+2e^{-\lambda T})}. \tag{2.23}$$

Differentiating the expected cost $C_p(T)$ with respect to T and putting it equal to 0, we obtain

$$\begin{aligned}4c_{e2}(1-e^{-\lambda T})+(4c_i+2c_r\lambda Te^{-\lambda T})(1-e^{-2\lambda T})\\+\left[2(c_i-c_{e2})+\frac{c_d}{\lambda}(1+2e^{-\lambda T})^2\right][1-(1+\lambda T)e^{-\lambda T}]\\-c_r(1+2e^{-\lambda T})[1-(1+2\lambda T)e^{-2\lambda T}]=6c_i+3c_{e2}.\end{aligned} \tag{2.24}$$

Letting denote the left-hand side of (2.24) by $Q_p(T)$,

$$Q_p(0)=0, \tag{2.25}$$

$$Q_p(\infty)=6c_i+2c_{e2}+\frac{1}{\lambda}c_d-c_r. \tag{2.26}$$

Thus, if $c_d/\lambda > c_{e2}+c_r$ then there exists exists a finite T_p^* $(0<T_p^*<\infty)$ which satisfies (2.24).

2.4 Numerical Examples

We compute numerically optimal intervals which minimize the expected costs per unit of time for each model. First, we consider the continuous comparison-checking and calculate the optimal interval T_c^*. Second, we consider the periodic comparison-checking and calculate the optimal interval T_p^*. All costs are normalized to c_i as a unit cost, *i.e.*, they are divided by c_i.

Table 2.1: Optimal diagnosis interval $\lambda T_c^* \times 10^5$ of continuous comparison-checking model for $c_d/(\lambda c_i)$ and c_r/c_i when $c_{e1}/(\lambda c_i) = 10^5$.

$c_d/(\lambda c_i) \times 10^{-7}$	$c_r/c_i \times 10^{-5}$		
	1	5	10
1	44.95	45.57	46.38
2	31.70	31.91	32.19
3	25.86	25.98	26.13
4	22.39	22.46	22.56
5	20.02	20.07	20.14
6	18.27	18.31	18.37
7	16.91	16.95	16.99
8	15.82	15.85	15.89
9	14.92	14.94	14.97
10	14.15	14.17	14.19

Table 2.1 gives $\lambda T_c^* \times 10^5$ which minimizes the expected cost $C_c(T)$ and satisfies (2.17) for $c_r/c_i \times 10^{-5} = 1, 5, 10$ and $c_d/(\lambda c_i) \times 10^{-7} = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ when $c_{e1}/(\lambda c_i) = 10^5$. It is shown that λT_c^* decreases as $c_d/(\lambda c_i)$ increases, and increases as c_r/c_i increases. This indicates, when the failure rate λ of each unit and the loss cost c_d increase, T_c^* decreases, *i.e.*, it is better to detect failures as early as possible. For example, $c_d/(\lambda c_i) \times 10^{-7} = 1$ and $c_r/c_i \times 10^{-5} = 1$, optimal $\lambda T_c^* \times 10^5 = 44.95$. That is, when the mean time of each unit is $1/\lambda = 3 \times 10^4$ hours (approximately 3.5 years), $c_{e1}/c_i = 1/3 \times 10$, $c_d/c_i = 1/3 \times 10^3$ and $c_r/c_i = 10^5$, optimal interval T_c^* is about 13.49 hours.

Table 2.2 gives $\lambda T_p^* \times 10^5$ which minimizes the expected cost $C_p(T)$ and satisfies (2.24) for $c_r/c_i \times 10^{-5} = 1, 5, 10$ and $c_d/(\lambda c_i) \times 10^{-7} = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ when $c_{e2}/c_i = 10^5$. This indicates that λT_p^* increases as $c_d/(\lambda c_i)$ increases. This shows the similar tendencies as the continuous comparison-checking model in Table 2.1.

Table 2.3 presents $\lambda T_p^* \times 10^5$ which minimizes the expected cost $C_p(T)$ and satisfies

Table 2.2: Optimal interval $\lambda T_p^* \times 10^5$ of periodic comparison-checking model for $c_d/(\lambda c_i)$ and c_r/c_i when $c_{e2}/c_i = 0.1$.

$c_d/(\lambda c_i) \times 10^{-7}$	$c_r/c_i \times 10^{-5}$		
	1	5	10
1	37.26	36.62	35.86
2	26.40	26.17	25.90
3	21.57	21.45	21.30
4	18.69	18.61	18.50
5	16.72	16.66	16.59
6	15.26	15.22	15.16
7	14.13	14.10	14.05
8	13.22	13.19	13.16
9	12.47	12.44	12.41
10	11.83	11.81	11.78

Table 2.3: Optimal interval $\lambda T_p^* \times 10^5$ of periodic comparison-checking model for c_{e2}/c_i and $c_d/(\lambda c_i)$ when $c_r/c_i = 5 \times 10^5$.

c_{e2}/c_i	$c_d/(\lambda c_i) \times 10^{-7}$		
	1	5	10
0.01	35.82	16.30	11.55
0.05	36.18	16.46	11.67
0.1	36.62	16.66	11.81
0.5	39.95	18.18	12.88

(2.24) for $c_{e2}/c_i = 0.01, 0.05, 0.1, 0.5$ and $c_d/(\lambda c_i) \times 10^{-7} = 1, 5, 10$ when $c_r/c_i = 5 \times 10^5$. This indicates that λT_p^* increases as c_{e2}/c_i increases. Thus, if the cost c_{e2}/c_i of comparison-checking is higher, it would be better to make its interval larger.

2.5 Conclusions

This chapter has considered a two-unit system which is checked by comparison-checking when it operates as a two-unit system, and by self-diagnosis after it is switched to a single-unit system. We have considered two models of continuous and periodic comparison-checking. First, the total expected cost of one cycle and the expected cost per unit of time have been analytically derived. Then, when the failure of each unit has an exponential distribution, we have obtained the expected costs and derived the optimal intervals of inspection which minimize the expected costs. It has been shown in numerical examples that the optimal interval is greatly affected by the loss cost for the time elapsed between a failure and its detection.

The studies of reliability and inspection for high reliable systems become more important subjects in real industries as systems become more complex and large. Further studies from such viewpoints are desirable, together with estimations of parameters for actual systems.

Appendix 2.1

Derivation of equation (2.9)

The expected costs in (2.6), (2.7) and (2.8) are calculated for each coefficient of costs: First, the term of c_{e2} is

$$2\sum_{m=1}^{\infty}\int_{mT}^{(m+1)T}\Big[\sum_{k=0}^{m-1}\int_{kT}^{(k+1)T}(k+1)dF(t_a)\Big]dF(t_b)+\sum_{k=0}^{\infty}\int_{kT}^{(k+1)T}(k+1)\big[F((k+1)T)-F(kT)\big]dF(t_a)$$

$$=\sum_{k=0}^{\infty}(k+1)\overline{F}((k+1)T)\big[\overline{F}(kT)-\overline{F}((k+1)T)\big]$$

$$+\sum_{k=0}^{\infty}(k+1)\overline{F}(kT)\big[\overline{F}(kT)-\overline{F}((k+1)T)\big]$$

$$=\sum_{k=0}^{\infty}\overline{F}(kT)^2. \tag{A.1}$$

Second, the term of c_i is

$$2\sum_{m=1}^{\infty}\int_{mT}^{(m+1)T}\Big[\sum_{k=0}^{m-1}\int_{kT}^{(k+1)T}(m-k)dF(t_a)\Big]dF(t_b)$$

$$=2\sum_{k=0}^{\infty}k\big[\overline{F}(kT)-\overline{F}((k+1)T)\big]\big[1-\overline{F}(kT)-\overline{F}((k+1)T)\big]$$

$$=2\sum_{k=0}^{\infty}\big[\overline{F}(kT)-\overline{F}(kT)^2\big]. \tag{A.2}$$

Finally, the term of c_d is the sum of the following three equation:

$$\sum_{k=0}^{\infty}\big[\overline{F}(kT)-\overline{F}((k+1)T)\big]\int_{kt}^{(k+1)T}[(k+1)T-t]\,dF(t)$$

$$+2\sum_{m=1}^{\infty}F(mT)\int_{mT}^{(m+1)T}[(m+1)T-t]\,dF(t)$$

$$+\sum_{k=0}^{\infty}\overline{F}((k+1)T)\int_{kT}^{(k+1)T}[(k+1)T-t]\,dF(t)$$

$$= \sum_{m=0}^{\infty} [1 + F(mT)] \int_{mT}^{(m+1)T} [(m+1)T - t] \, dF(t)$$

$$= \sum_{m=0}^{\infty} [1 + F(mT)] \int_{mT}^{(m+1)T} [F(t) - F(mT)] \, dt. \tag{A.3}$$

By summing (A.1), (A.2) and (A.3) and adding the maintenance cost c_r, the total expected cost of one cycle is

$$c_r + (c_{e2} - c_i) \sum_{m=0}^{\infty} \overline{F}(mT)^2 + c_i \sum_{m=0}^{\infty} \left[1 - F(mT)^2\right]$$
$$+ c_d \sum_{m=0}^{\infty} \left[1 + F(mT)\right] \int_{mT}^{(m+1)T} \left[F(t) - F(mT)\right] dt.$$

Chapter 3

Optimal Periodic Inspection Policies for a System with Self-Testing

This chapter considers optimal inspection policies for a system with self-testing: The system with self-testing can detect any failure without performing external inspection. However, some failures might not be detected rapidly. Therefore, it would be necessary to perform the inspection periodically. The total expected cost and expected cost per unit of time are obtained. and optimal policies which minimize them are analytically derived. Numerical examples are given when the failure time is exponential.

3.1 Introduction

We consider a system such as digital control devices for aircraft engines which have sequential input and output codes. We suppose that the system has the property of self-testing: If the system has at least one input code which gives some output codes in outside of an assumed output code space, when there exist any failure in an assumed failure set, then the system has the property of self-testing. Therefore, any failure in a failure set can be detected without external inspection by checking, whichever the output codes are in an assumed space or not. Thus, the system with self-testing can detect any failure during the normal operation [Lala (2001)].

However, even if the system has input codes to detect some failures, they might not be readily inputed to the system. Therefore, some failures might not be detected rapidly by self-testing. Hence, to detect failures early and surely, it would be necessary to perform external inspection such as inputting a set of test codes at periodic times. In this case, if the system fails, then its failure is detected by self-testing or the next periodic inspection, whichever occur first. However, it might incur much loss cost to perform periodic inspection so frequently [O'Connor (2001), Savir *et al.* (1984a, 1984b), Shedletsky and McCluskey (1975a, 1975b), Parker and McCluskey (1975)].

In general, to design high reliable systems it is required to improve the property of self-checking, where the self-checking involves properties of fault-secure and self-testing. Fault-secure means the property that a failed system outputs either correct codes or codes which is not in an assumed output code space. That is, the system with fault-secure does not output codes which are in an assumed code space and incorrect for the result of input codes. In this chapter, we consider only the property of self-testing and do not consider the property of self-checking, because we want to form stochastic models from the viewpoint that the system can detect failures by itself without external inspection, and fault-secure has no relation with these models [Lala (2001)].

In this chapter, it is assumed that the time from the occurrence of failure to its detection by self-testing has a probability distribution [Savir *et al.* (1984a, 1984b)]. When failures are detected, the system becomes like new and starts to operate again. Then, introducing the loss cost for the time elapsed between a failure and its detection, the total expected cost until the detection of failure is obtained. Optimal intervals of periodic inspections which minimize the total expected cost and the expected cost per

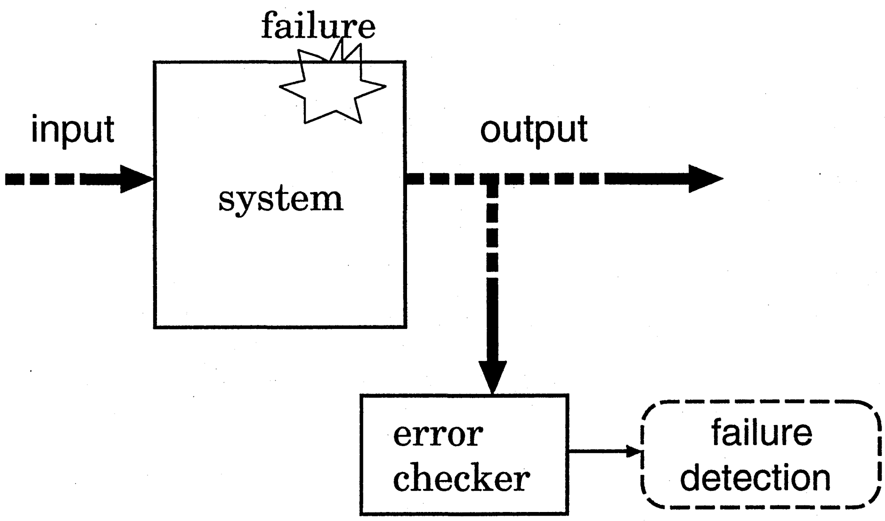

Figure 3.1: System with self-testing.

unit of time are analytically derived, using the self-detection rate. It is of great interest that the self-detection rate plays an important role for analyzing such optimal policies. Further, we consider the case where there exist some failures which can not be detected by self-testing with probability p. Finally, numerical examples are given when both times of failure and its detection by self-testing are exponential. Moreover, we treat a sequential inspection policy where inspection times are not periodic and made at successive times.

3.2 Model and Assumptions

3.2.1 Periodic inspection model

Consider periodic inspection policies for a system with self-testing, which can detect any failure. For this model, we define the following assumptions:

(i) The system is checked at periodic times kT $(k = 1, 2, \dots)$ by inspection. Thus, when the system fails, its failure will be detected by self-testing or at the next periodic inspection, whichever occurs first.

(ii) The failure time distribution has a general distribution $F(t)$ with finite mean $1/\lambda$, where $\overline{F}(t) \equiv 1 - F(t)$.

(iii) If we do not consider the detection of failure by periodic inspection, then the time from a failure to its detection by self-testing has a general distribution $G(x)$ with mean $1/\mu$ ($\mu > \lambda$), independent of the failure time, where $1/\mu$ might be infinity.

(iv) A cost c_i is the cost for one check by periodic inspection, and c_d is the loss cost per unit of time for the time elapsed between a failure and its detection by self-testing or periodic inspection, whichever occurs first. A cost c_r is the replacement or maintenance cost for a failed system.

Figure 3.2 shows the processes of the system with self-testing: The horizontal axises present the process of time. This system is checked at periodic times kT ($k = 1, 2, \dots$) by inspection, which incurs the loss cost c_i for every one check. When the system fails at time t ($kT < t \leq (k+1)T$), the upper side shows the case where its failure is detected at time $t + x$ ($< (k+1)T$) by self-testing, and the lower side shows the case where its failure is detected at time $(k+1)T$ before the self-testing by periodic inspection, *i.e.*, $(k+1)T < t + x$.

We define one cycle as the time from the beginning of system operation to the detection of its failure. Then, the mean time of one cycle is

$$A(T) = \sum_{k=0}^{\infty} \int_{kT}^{(k+1)T} \left[\int_0^{(k+1)T-t} (t+x)\, dG(x) + (k+1)T\,\overline{G}((k+1)T - t) \right] dF(t)$$

$$= \frac{1}{\lambda} + \sum_{k=0}^{\infty} \int_0^T \left[\overline{F}(kT) - \overline{F}((k+1)T - x) \right] \overline{G}(x)\, dx, \tag{3.1}$$

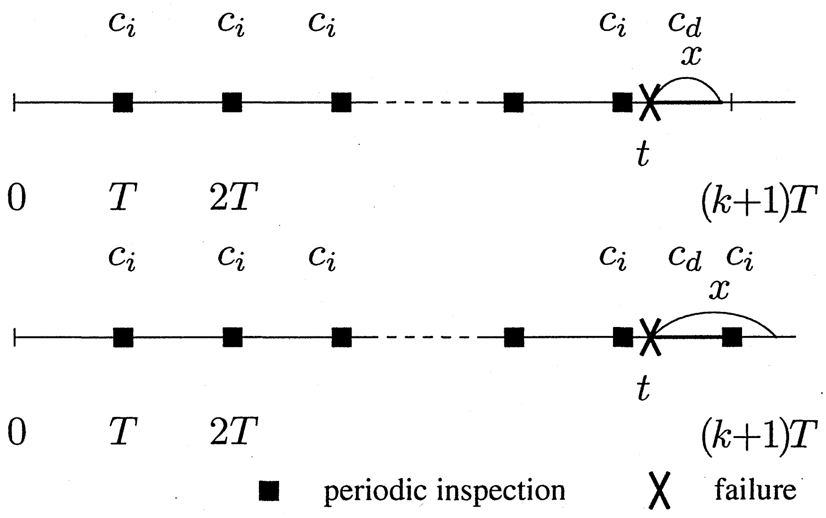

Figure 3.2: Processes of system with self-testing.

where, in general, $\overline{\Phi}(t) \equiv 1 - \Phi(t)$. In a similar way, the total expected cost of one cycle is given by

$$\begin{aligned}B(T) = \sum_{k=0}^{\infty} \int_{kT}^{(k+1)T} \Bigg[\int_0^{(k+1)T-t} (c_i k + c_d x)\, dG(x) \\ + \big\{c_i(k+1) + c_d[(k+1)T - t]\big\}\overline{G}((k+1)T - t)\Bigg] dF(t) + c_r \\ = c_i \sum_{k=0}^{\infty} \left\{ \overline{F}(kT) - \int_0^T \Big[\overline{F}(kT) - \overline{F}((k+1)T - x)\Big] dG(x) \right\} \\ + c_d \sum_{k=0}^{\infty} \int_0^T \Big[\overline{F}(kT) - \overline{F}((k+1)T - x)\Big] \overline{G}(x)\, dx + c_r. \end{aligned} \tag{3.2}$$

It is evident that

$$B(0) \equiv \lim_{T \to 0} B(T) = \infty,$$

$$B(\infty) \equiv \lim_{T \to \infty} B(T) = \frac{c_d}{\mu} + c_r.$$

Thus, there exists an optimal time T^* $(0 < T^* \leq \infty)$ which minimizes $B(T)$.

In particular, when $\overline{G}(x) \equiv 1$, *i.e.*, any failure is detected only by periodic inspection, equations (3.1) and (3.2) are simplified as

$$A(T) = T\sum_{k=0}^{\infty}\overline{F}(kT),$$

$$B(T) = (c_i + c_d T)\sum_{k=0}^{\infty}\overline{F}(kT) - \frac{c_d}{\lambda} + c_r,$$

which agree with the results of the simple periodic inspection policy [Barlow and Proschan (1965)].

Therefore, the expected cost per unit of time is, from (3.1) and (3.2),

$$C(T) \equiv \frac{B(T)}{A(T)}$$

$$= c_d + \frac{c_i\sum_{k=0}^{\infty}\left\{\overline{F}(kT) - \int_0^T[\overline{F}(kT) - \overline{F}((k+1)T - x)]\,dG(x)\right\} - \frac{c_d}{\lambda} + c_r}{\sum_{k=0}^{\infty}\int_0^T[\overline{F}(kT) - \overline{F}((k+1)T - x)]\overline{G}(x)\,dx + 1/\lambda}. \tag{3.3}$$

Evidently,

$$C(0) \equiv \lim_{T\to 0} C(T) = \infty,$$

$$C(\infty) \equiv \lim_{T\to\infty} C(T) = \frac{c_d/\mu + c_r}{1/\lambda + 1/\mu}.$$

Thus, there exists an optimal time T^* $(0 < T^* \leq \infty)$ which minimizes $C(T)$.

3.2.2 Sequential inspection model

Consider the sequential inspection policy for a system with self-testing, where intervals of inspection are not periodic. For this model, we define the following assumptions:

(i) The system is checked at successive times T_k $(k = 1, 2, \dots)$, where $T_0 \equiv 0$. Thus, when the system fails, its failure will be detected by self-testing or at the next inspection, whichever occurs first.

(ii) A cost c_i is the cost for one check at times T_k $(k = 1, 2, \dots)$ by inspection, and c_d is the loss cost per unit of time for the time elapsed between a failure and its detection at the next checking time, and c_r is the replacement or maintenance cost for a failed system.

(iii) Make the same assumptions as (ii) and (iii) of the previous model.

Then, in a similar way of obtaining (3.3), the expected cost of one cycle is given by

$$\mathbf{C}(T_1, T_2, \dots) \equiv$$

$$\sum_{k=0}^{\infty} \int_{T_k}^{T_{k+1}} \left\{ \int_0^{T_{k+1}-t} (c_i k + c_d x) dG(x) + [c_i(k+1) + c_d(T_{k+1} - t)]\overline{G}(T_{k+1} - t) \right\} dF(t)$$

$$= c_i \sum_{k=0}^{\infty} \left\{ \overline{F}(T_k) - \int_0^{T_{k+1}-T_k} [\overline{F}(T_k) - \overline{F}(T_{k+1} - x)] dG(x) \right\}$$

$$+ c_d \sum_{k=0}^{\infty} \int_0^{T_{k+1}-T_k} [\overline{F}(T_k) - \overline{F}(T_{k+1} - x)]\overline{G}(x) dx. \tag{3.4}$$

3.3 Optimal Inspection Policy

3.3.1 Optimal policy for total expected cost

We seek an optimal time T^* which minimizes the total expected cost $B(T)$ in (3.2), when the failure time is exponential, *i.e.*, $F(t) = 1 - e^{-\lambda t}$. Then, the total expected cost $B(T)$ is rewritten as

$$B(T) = \frac{c_i \left[1 - \int_0^T (1 - e^{-\lambda(T-x)})\, dG(x) \right] + c_d \int_0^T (1 - e^{-\lambda(T-x)})\overline{G}(x)\, dx}{1 - e^{-\lambda T}} + c_r. \tag{3.5}$$

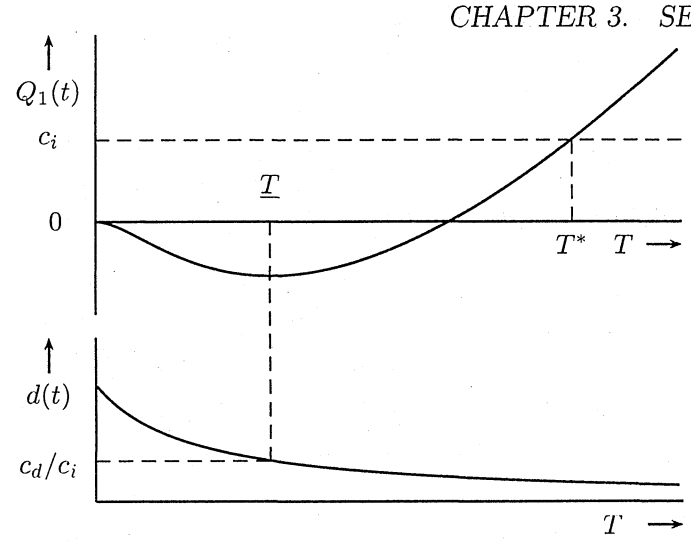

Figure 3.3: Relationship between self-detection rate $d(T)$ and function $Q_1(T)$.

Differentiating $B(T)$ with respect to T and putting it equal to zero, we have

$$c_d \int_0^T (e^{\lambda x} - 1)\overline{G}(x)\, dx - c_i \int_0^T (e^{\lambda x} - 1)\, dG(x) = c_i. \tag{3.6}$$

Letting denote the left-hand side of (3.6) by $Q_1(T)$,

$$Q_1(0) \equiv \lim_{T\to 0} Q_1(T) = 0,$$

$$Q_1(\infty) \equiv \lim_{T\to\infty} Q_1(T) = c_d \int_0^\infty (e^{\lambda x} - 1)\overline{G}(x)\, dx - c_i \int_0^\infty (e^{\lambda x} - 1)\, dG(x),$$

$$Q_1'(T) = (e^{\lambda T} - 1)\overline{G}(T)c_i\left[\frac{c_d}{c_i} - d(T)\right],$$

where $d(t) \equiv g(t)/\overline{G}(t)$ and $g(t)$ is a density of G, and $d(t)\,dt$ represents the probability that when the system has failed, its failure will be detected during $(t, t+dt)$ by self-testing. We call $d(t)$ *self-detection rate.* It would be practically estimated that $d(t)$ is decreasing. In this case, if $d(T) \geq c_d/c_i$ then $Q_1(T)$ decreases, and conversely, if $d(T) < c_d/c_i$ then $Q_1(T)$ increases (see Figure 3.3).

Therefore, we have the following optimal policy when $d(t)$ is decreasing:

(i) If $Q_1(\infty) > c_i$ then there exists a finite and unique T^* $(0 < T^* < \infty)$ which satisfies (3.6).

(ii) If $Q_1(\infty) \leq c_i$ then $T^* = \infty$.

It is noted that if there exists a solution $\underline{T}$ to satisfy $d(T) = c_d/c_i$ then $T^* > \underline{T}$.

In particular, when $G(x) = 1 - e^{-\mu x}$ $(\mu > \lambda)$, *i.e.*, $d(t) = \mu$, equation (3.6) is rewritten as

$$(c_d - \mu c_i)\left(\frac{1 - e^{-(\mu-\lambda)T}}{\mu - \lambda} - \frac{1 - e^{-\mu T}}{\mu}\right) = c_i. \tag{3.7}$$

Letting

$$L(T) \equiv \frac{1 - e^{-(\mu-\lambda)T}}{\mu - \lambda} - \frac{1 - e^{-\mu T}}{\mu},$$

it is strictly increasing from 0 to $\lambda/[\mu(\mu-\lambda)]$. Therefore, we have the following optimal policy:

(i) If $\lambda c_d/\mu^2 > c_i$ then there exists a finite unique T^* $(0 < T^* < \infty)$ which satisfies (3.7).

(ii) if $\lambda c_d/\mu^2 \leq c_i$ then $T^* = \infty$.

Furthermore, we consider the case where there exist some failures which can not be detected by self-testing. That is, some failures, which are not in an assumed set detected by self-testing, might occur. In the design of a complex system, it would be difficult to design a system which can detect any failure. Therefore, it is realistic to design a system which have priority over high detection rate at early times more than the property to detect any failure. It is assumed that p $(0 < p < 1)$ is probability that the failure can be detected by self-testing. On the other hand, $q \equiv 1 - p$ is probability that it can not be detected by self-testing, *i.e.*, it can be detected only by periodic inspection. From, the above discussions, it is reasonable to assume that $G(x) = p(1 - e^{-\mu x})$, and hence, the self-detection rate is given by

$$d(x) = \frac{p\mu e^{-\mu x}}{q + pe^{-\mu x}},$$

which is a decreasing function from $p\mu$ to 0. If $p = 1$ then this model corresponds to the case where $G(x)$ is a standard exponential distribution, and if $p = 0$ then this model corresponds to only the periodic inspection one.

In this case, equation (3.6) is rewritten by

$$p(c_d - \mu c_i)\left(\frac{1 - e^{-(\mu-\lambda)T}}{\mu - \lambda} - \frac{1 - e^{-\mu T}}{T}\right) + q\frac{c_d}{\lambda}(e^{\lambda T} - \lambda T - 1) = c_i. \tag{3.8}$$

Letting denote the left-hand side of (3.8) by $Q_2(T)$,

$$Q_2(\infty) \equiv \lim_{T\to\infty} Q_2(T) = \infty,$$
$$Q_2'(T) = (e^{\lambda T} - 1)[(c_d - \mu c_i)e^{-\mu T} + qc_d].$$

Thus, if $c_d/\mu > c_i$ then $Q_2'(T) > 0$, and hence, $Q_2(T)$ increases strictly from 0 to ∞. Therefore, there exists a finite and unique T^* $(0 < T^* < \infty)$ which satisfies (3.8).

3.3.2 Optimal policy for expected cost per unit of time

Consider the problem of minimizing the expected cost $C(T)$ in (3.3). In particular, when $F(t) = 1 - e^{-\lambda t}$, the expected cost is

$$C(T) = c_d + \frac{c_i\left[1 - \int_0^T (1 - e^{-\lambda(T-x)})\, dG(x)\right] - \left(\frac{c_d}{\lambda} - c_r\right)(1 - e^{-\lambda T})}{\int_0^T (1 - e^{-\lambda(T-x)})\overline{G}(x)\, dx + \frac{1}{\lambda}(1 - e^{-\lambda T})}. \tag{3.9}$$

Differentiating $C(T)$ with respect to T and putting it equal to zero,

$$(c_d - \lambda c_r)\int_0^T (e^{\lambda x} - 1)\overline{G}(x)\,dx - c_i\left[\int_0^T (e^{\lambda x} - 1)\,dG(x) + \int_0^T \lambda e^{\lambda x}\overline{G}(x)\,dx\right.$$
$$\left. + \int_0^T \lambda e^{\lambda x}\,dG(x)\int_0^T (1 - e^{-\lambda(T-x)})\overline{G}(x)\,dx - \int_0^T \lambda e^{\lambda x}\overline{G}(x)\,dx\int_0^T (1 - e^{-\lambda(T-x)})\,dG(x)\right]$$
$$= c_i. \tag{3.10}$$

Letting denote the left-hand side of $Q_3(T)$,

$$Q_3(0) \equiv \lim_{T\to 0} Q_3(T) = 0,$$
$$Q_3(\infty) \equiv \lim_{T\to\infty} Q_3(T)$$
$$= (c_d - \lambda c_r)\int_0^\infty (e^{\lambda x} - 1)\overline{G}(x)\,dx - c_i\left\{\int_0^\infty (e^{\lambda x} - 1)\,dG(x) + \frac{\lambda}{\mu}\int_0^\infty e^{\lambda x}\,dG(x)\right\},$$
$$Q_3'(T) = (e^{\lambda T} - 1)\overline{G}(T)c_i\left[\frac{c_d - \lambda c_r}{c_i} - d(T) - \frac{\lambda}{1 - e^{-\lambda T}}\right]$$
$$+ \lambda e^{\lambda T}\overline{G}(T)c_i\int_0^T (1 - e^{-\lambda(T-x)})\overline{G}(x)\,dx\left[\frac{\int_0^T (1 - e^{-\lambda(T-x)})\,dG(x)}{\int_0^T (1 - e^{-\lambda(T-x)})\overline{G}(x)\,dx} - d(T)\right].$$

When the self-detection rate $d(t)$ is decreasing, *i.e.*, $d(T) \le g(x)/\overline{G}(x)$ for $0 \le x \le T$, we have

$$\frac{\int_0^T (1 - e^{-\lambda(T-x)})\,dG(x)}{\int_0^T (1 - e^{-\lambda(T-x)})\overline{G}(x)\,dx} \ge d(T). \tag{3.11}$$

Thus, $Q_3(T)$ decreases at first, and after that, increases to $Q_3(\infty)$. Therefore, if $Q_3(\infty) > c_i$ then there exists a finite and unique T^* $(0 < T^* < \infty)$ which minimizes $C(T)$.

In particular, when $G(x) = 1 - e^{-\mu x}$, equation (3.10) is rewritten as

$$(c_d - \lambda c_r - \mu c_i)\left(\frac{1 - e^{-(\mu-\lambda)T}}{\mu - \lambda} - \frac{1 - e^{-\mu T}}{\mu}\right) - c_i\frac{\lambda(1 - e^{-(\mu-\lambda)T})}{\mu - \lambda} = c_i. \tag{3.12}$$

Further, if $1/\mu \to \infty$ then equation is simplified as

$$(c_d - \lambda c_r)\frac{1-(1+\lambda T)e^{-\lambda T}}{\lambda} = c_i. \tag{3.13}$$

Let denote the left-hand side of (3.12) by $Q_4(T)$. Then, we have

$$Q_4(\infty) \equiv \lim_{T\to\infty} Q_4(T) = (c_d - \lambda c_r - \mu c_i)\frac{\lambda}{\mu(\mu-\lambda)},$$
$$Q_4'(T) = e^{-(\mu-\lambda)T}[(c_d - \lambda c_r - \mu c_i)(1-e^{-\lambda T}) - \lambda c_i].$$

Thus, $Q_4(T)$ starts from 0 and decreases for a while, and after that, increases strictly to $Q_4(\infty)$ for $c_d - \lambda c_r - \mu c_i > 0$.

Therefore, we have the optimal policy:

(iii) If $(c_d - \lambda c_r - \mu c_i)\{\lambda/[\mu(\mu-\lambda)]\} > c_i$ then there exists a finite and unique T^* which satisfies (3.12).

(iv) If $(c_d - \lambda c_r - \mu c_i)\{\lambda/[\mu(\mu-\lambda)]\} \le c_i$ then $T^* = \infty$.

Furthermore, we consider the case where $G(x) = p(1-e^{-\mu x})$ $(0 < p < 1)$. Then, the expected cost $C(T)$ in (3.3) is

$$C(T) = \frac{c_i\left[1-\mu p\left(\frac{1-e^{-\mu T}}{\mu} - \frac{e^{-\lambda T}-e^{-(\mu-\lambda)T}}{\mu-\lambda}\right)\right] - \left(\frac{c_d}{\lambda}-c_r\right)(1-e^{-\lambda T})}{p\left(\frac{1-e^{-\mu T}}{\mu} + \frac{1-e^{-\lambda T}}{\lambda} - \frac{e^{-\lambda T}-e^{-(\mu-\lambda)T}}{\mu-\lambda}\right) + qT} + c_d. \tag{3.14}$$

Equation (3.10) is

$$\begin{aligned} &p\left[(c_d - \lambda c_r - \mu c_i)\left(\frac{1-e^{-(\mu-\lambda)T}}{\mu-\lambda} - \frac{1-e^{-\mu T}}{\mu}\right) - \frac{c_i\lambda}{\mu-\lambda}(1-e^{-(\mu-\lambda)T})\right] \\ &+ q\left[\frac{c_d-\lambda c_r}{\lambda}(e^{\lambda T}-\lambda T-1) - c_i(e^{\lambda T}-1)\right] \\ &- pq\left[\frac{\lambda\mu T}{\mu-\lambda}(1-e^{-(\mu-\lambda)T}) - (e^{\lambda T}-1)(1-e^{-\mu T})\right] = c_i. \end{aligned} \tag{3.15}$$

Letting denote the left-hand side of (3.15) by $Q_5(T)$, we have

$$\begin{aligned} Q_5(\infty) &\equiv \lim_{T\to\infty} Q_5(T) \\ &= p\left[(c_d - \lambda c_r - \mu c_i)\left(\frac{1}{\mu-\lambda} - \frac{1}{\mu}\right) - \frac{\lambda c_i}{\mu-\lambda}\right] \\ &\quad + \lim_{T\to\infty} q\left\{\left[\frac{c_d - \lambda(c_r + qc_i)}{\lambda}\right](e^{\lambda T} - 1) - \left(c_d - \lambda c_r + \frac{\lambda\mu p c_i}{\mu-\lambda}\right)T\right\}. \end{aligned}$$

Therefore, if $c_d - \lambda(c_r + qc_i) > 0$ then $Q_5(\infty) = \infty$ and there exists optimal T^* $(0 < T^* < \infty)$ which minimizes $C(T)$. Obviously, if $\lambda(c_d - \lambda c_r - \mu c_i)/\mu^2 > c_i$ then $c_d - \lambda(c_r + qc_i) > 0$.

3.3.3 Optimal policy for sequential inspection

Consider the problem of minimizing the expected cost $\mathbf{C}(T_1, T_2, \dots)$ in (3.4). Differentiating with T_k and putting it to zero, we have

$$\begin{aligned} &\int_0^{T_{k+1}-T_k} \overline{G}(x)dx + \frac{c_i}{c_d}\overline{G}(T_{k+1} - T_k) \\ &= \frac{1}{f(T_k)}\left[\int_0^{T_k - T_{k-1}} f(T_k - x)\overline{G}(x)dx - \frac{c_i}{c_d}\int_0^{T_k - T_{k-1}} f(T_k - x)dG(x)\right] \quad (k = 1, 2, \dots). \end{aligned} \tag{3.16}$$

Note that if $\overline{G}(x) \equiv 1$, *i.e.*, the failure can not be detected by self-testing, then equation (3.16) is rewritten as

$$T_{k+1} - T_k = \frac{F(T_k) - F(T_{k-1})}{f(T_k)} - \frac{c_i}{c_d},$$

which agrees with the result of standard inspection policy [Barlow and Proschan (1965)].

In particular, when $F(t) = 1 - e^{-\lambda t}$, equation (3.16) is

$$\int_0^{T_{k+1}-T_k} \overline{G}(x)dx + \frac{c_i}{c_d}\overline{G}(T_{k+1} - T_k)$$
$$= \int_0^{T_k - T_{k-1}} e^{\lambda x}\overline{G}(x)dx - \frac{c_i}{c_d}\int_0^{T_k - T_{k-1}} e^{\lambda x} dG(x) \qquad (k = 1, 2, \dots). \qquad (3.17)$$

Then, $T_k^* = kT_1^*$ $(k = 1, 2, \dots)$ is shown in Appendix 3.1. That is, when the failure time has an exponential distribution, we should check the system by periodic inspection. The reason is that when the failure time has an exponential distribution, the system does not degrade with time and hence, it is not necessary to shorten the intervals of inspection with time.

On the other hand, when $G(x) = 1 - e^{-\mu x}$, equation (3.16) is rewritten as

$$\frac{1}{f(T_k)}\int_0^{T_k - T_{k-1}} f(T_k - x)\mu e^{-\mu x} dx - [1 - e^{-\mu(T_{k+1} - T_k)}] = \frac{c_i}{\frac{c_d}{\mu} - c_i} \qquad (k = 1, 2, \dots), \qquad (3.18)$$

i.e.,

$$T_{k+1} = T_k - \frac{1}{\mu}\log\left(\frac{1}{1 - \mu\frac{c_i}{c_d}} - \frac{1}{f(T_k)}\int_0^{T_k - T_{k-1}} f(T_k - x)\mu e^{-\mu x} dx\right). \qquad (3.19)$$

To obtain T_k^* $(k = 1, 2, \dots)$ which satisfies (3.19), we use the Barlow's algorithm [Barlow and Proschan (1965)] as follows:

1. Choose T_1 at random.

2. Compute $T_2, T_3, \cdots$ recursively from (3.18).

3. If any $\delta_k > \delta_{k-1}$, reduce T_1 and repeat where $\delta_k \equiv T_{k+1} - T_k$. If any $\delta_k < 0$, increase T_1 and repeat.

4. Continue until $T_1 < T_2 < \cdots$ are determined to the degree of accuracy required.

Table 3.1: Optimal interval T^* to minimize $B(T)$ for $1/\mu$ and c_d/c_i when $1/\lambda = 3\times 10^5$.

$1/\mu$	c_d/c_i		
	100	250	500
20	∞	∞	∞
30	∞	∞	68.68
40	∞	107.71	52.21
50	∞	80.77	46.70
60	194.11	71.33	43.86
70	144.80	66.32	42.12
80	126.44	63.17	40.93
90	116.27	61.01	40.07
100	109.73	59.42	39.42
∞	77.46	48.99	34.64

3.4 Numerical Examples

We compute numerical examples for each model when $F(t) = 1 - e^{-\lambda t}$. First, we calculate optimal interval which minimizes the total expected cost $B(T)$ in (3.5). Second, we calculate optimal interval which minimizes the expected cost $C(T)$ in (3.9) when $G(x) = 1 - e^{-\mu x}$. Third, we calculate T^* which minimizes the expected cost $C(T)$ in (3.14) when $G(x) = p(1 - e^{-\mu x})$. Finally, we compute the sequential scheduled times which minimize the expected cost in (3.19) when $F(t) = 1 - e^{-\lambda t^m}$. The cost c_d is normalized to c_i as a unit cost, *i.e.*, it is divided by c_i.

Table 3.1 gives the optimal interval T^* which minimizes the total expected cost $B(T)$ in (3.5), when $G(x) = 1 - e^{-\mu x}$, and satisfies (3.6) for $1/\mu = 20$, 30, 40, 50, 60, 70, 80, 90, 100, ∞, and $c_d/c_i = 100$, 250, 500 when $1/\lambda = 3 \times 10^5$. This indicates that optimal interval T^* decreases when $1/\mu$ increases, and tends to a fixed value as $1/\mu$ or c_d/c_i goes to infinity, which is a solution of (3.13). From the optimal policy, if $1/\mu \leq \sqrt{c_i/(\lambda c_d)}$ then $T^* = \infty$. For example, when $c_d/c_i = 100$, if $1/\mu < \sqrt{3,000}$ then

Table 3.2: Optimal interval T^* to minimize $C(T)$ for $1/\mu$ and c_d/c_i when $G(x) = 1 - e^{-\mu x}$.

$1/\mu$	c_d/c_i		
	100	250	500
20	∞	∞	∞
30	∞	∞	68.69
40	∞	107.75	52.21
50	∞	80.79	46.71
60	194.32	71.34	43.87
70	144.89	66.33	42.12
80	126.50	63.18	40.93
90	116.34	61.02	40.07
100	109.78	59.43	39.42
∞	77.48	49.00	34.64

$T^* = \infty$. This shows that if a failure is detected early and surely by self-testing, then it is not necessary to perform periodic inspection.

Table 3.2 gives the optimal interval T^* which minimizes the expected cost $C(T)$ in (3.9) when $G(x) = 1 - e^{-\mu x}$, and satisfies (3.12) for $1/\mu = 20, 30, 40, 50, 60, 70, 80, 90, 100, \infty$, and $c_d/c_i = 100, 250, 500$ when $1/\lambda = 3 \times 10^5$, $c_r/c_i = 10^4$. From the optimal policy, if $1/\mu \le 2/[-\lambda + \sqrt{\lambda^2 + 4\lambda(c_d - \lambda c_r)/c_i}]$ then $T^* = \infty$. Optimal interval T^* in Table 3.2 tends to be a little greater than that in Table 3.1.

Table 3.3 shows the optimal interval T^* which minimizes the expected cost $C(T)$ when $G(x) = p(1 - e^{-\mu x})$, and satisfies (3.15) for $1/\mu = 20, 30, 40, 50, 60, 70, 80, 90, 100$ and $p = 0.9, 0.5, 0.2, 0.0$ when $1/\lambda = 3 \times 10^5$, $c_d/c_i = 100$ and $c_r/c_i = 10^4$. For example, if $1/\mu = 40$ and $p = 0.5$ then $T^* = 98.72$. This indicates that T^* decreases as p decreases from 1 to 0, where note that if $p = 0$ then the system can not detect its failure by self-testing.

Table 3.4 shows the optimal inspection time which minimizes the expected cost

Table 3.3: Optimal interval T^* to minimize $C(T)$ for $1/\mu$ and p when $G(x) = p(1-e^{-\mu x})$ and $c_d/c_i = 100$.

$1/\mu$	p			
	0.9	0.5	0.2	0.0
20	229.82	105.98	85.55	77.48
30	209.62	102.31	84.59	77.48
40	181.48	98.72	83.72	77.48
50	153.43	95.68	82.98	77.48
60	133.49	93.24	82.38	77.48
70	120.87	91.29	81.88	77.48
80	112.65	89.73	81.47	77.48
90	106.98	88.47	81.12	77.48
100	102.85	87.43	80.83	77.48

Table 3.4: Optimal times $T_k^* - T_{k-1}$ $(k = 1, 2, \ldots, 12)$ for m to minimize $\mathbf{C}(T_1, T_2, \ldots)$ when $G(x) = 1 - e^{-\mu x}$, $F(t) = 1 - e^{-\lambda t^m}$ and $c_d/c_i = 100$.

k	m		
	1.0	1.5	2.0
1	109.78	40.90	23.08
2	109.78	26.51	11.32
3	109.78	23.48	9.40
4	109.78	21.78	8.36
5	109.78	20.61	7.68
6	109.78	19.74	7.17
7	109.78	19.05	6.78
8	109.78	18.47	6.46
9	109.78	18.00	6.20
10	109.78	17.57	5.97
11	109.78	17.21	5.77
12	109.78	16.88	5.60

$\mathbf{C}(T_1, T_2, \dots)$ in (3.18) when $F(t) = 1 - e^{-\lambda t^m}$ for $m = 1,\ 1.5,\ 2$ when $1/\lambda = 3 \times 10^5$ and $c_d/c_i = 100$. For example, when $m = 1.5$ then the system is checked with intervals 40.90, 26.51, 23.48, and so on. This indicates that T_k^* decreases as m increases. When $m = 1.0$, these are equal to the result of periodic inspection.

3.5 Conclusions

We have proposed the optimal testing policies for a system with self-testing: Using the theory of inspection policy in reliability, we have derived the total expected cost until the detection of failure and the expected cost per unit of time, and discussed analytically the optimal inspection policies which minimize them. It has been shown that the self-detection rate plays an important role for deriving optimal policies. For designing a good performance of the system, it would be necessary to increase the self-detection rate by improving the property of self-testing.

Appendix 3.1

Prove that $T_k^* = kT_1^*$ $(k = 1, 2, \ldots)$ in equation (3.17)

Let T_1^* be a solution to satisfy

$$\int_0^{T_1} \overline{G}(x)dx + \frac{c_i}{c_d}\overline{G}(T_1) = \int_0^{T_1} e^{\lambda x}\overline{G}(x)dx - \frac{c_i}{c_d}\int_0^{T_1} e^{\lambda x}dG(x).$$

Further, when $k = 1$ in (3.17),

$$\int_0^{T_2-T_1} \overline{G}(x)dx + \frac{c_i}{c_d}\overline{G}(T_2 - T_1) = \int_0^{T_1} e^{\lambda x}\overline{G}(x)dx - \frac{c_i}{c_d}\int_0^{T_1} e^{\lambda x}\overline{G}(T_1).$$

Thus, we have

$$\int_0^{T_2-T_1} \overline{G}(x)dx + \frac{c_i}{c_d}\overline{G}(T_2 - T_1) = \int_0^{T_1} \overline{G}(x)dx + \frac{c_i}{c_d}\overline{G}(T_1),$$

and hence, $T_2^* = 2T_1^*$.

Similarly, it can be easily proved that $T_k^* = kT_1^*$ $(k = 1, 2, \ldots)$.

Chapter 4

Optimal Maintenance and Inspection Policies for a Finite Interval

This chapter considers optimal policies for maintenance and inspection models for a finite interval. It would be important to consider practically some maintenance policies for a finite span, because the working times of most units are finite in actual fields. We convert the usual replacement models to maintenance models for a finite interval, and derive optimal policies for each model, using the partition method. Further, we show how to compute numerically optimal checking times of a finite inspection model. Numerical examples are given for each model.

4.1 Introduction

This chapter considers optimal policies for maintenance and inspection models where a unit has to operate for a finite interval. Practically, the working times of most units are finite in actual fields.

There have little papers treated with replacements for a finite time span. Barlow and Proschan (1965) derived the optimal sequential policy for age replacement for a

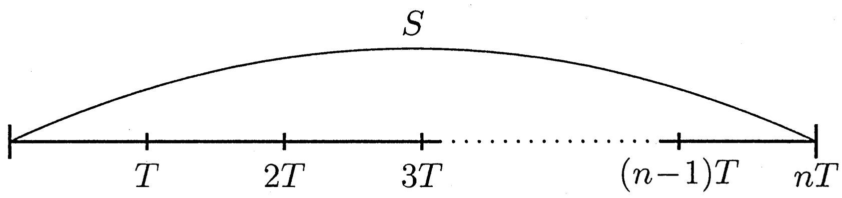

Figure 4.1: Finite time S with n periodic intervals.

finite interval. Christer (1978) and Ansell *et al.* (1984) gave the asymptotic costs of age replacement for a finite interval. Nakagawa *et al.* (2004) considered the inspection model for a finite working time and gave the optimal policy, by partitioning the working time into equal parts.

This chapter proposes modified replacement policies which convert three usual models of periodic replacement with minimal repair, block replacement and simple replacement to replacements of units for a finite interval. The optimal policies of three replacements are analytically derived by using the partition method in Nakagawa *et al.* (2004). Further, it is shown that all equations of three replacements are written on general forms. Next, we consider the sequential inspection policy in which a unit is checked at successive times for a finite interval, and show how to compute optimal checking times numerically.

4.2 Replacement Policies

We suppose that a unit has to be operating for a finite interval $[0, S]$, *i.e.*, its working time is given by a specified value S. To maintain the unit, an interval S is partitioned equally into n parts in which it is replaced at periodic times kT $(k = 1, 2, \ldots, n)$ (see Figure 4.1), where $nT \equiv S$. Then, we consider the replacement with minimal repair at failure, the block replacement and the simple replacement [Nakagawa pp. 367–395 (2003)].

4.2.1 Periodic replacement with minimal repair

We consider the model where a unit is replaced with minimal repair as follows:

(i) The unit is replaced at periodic times kT $(k = 1, 2, \ldots, n)$ and any unit is as good as new with each replacement. When the unit fails between replacements, only minimal repair is made, and hence, its failure rate remains undisturbed by any repair of failures. It is assumed that the repair and replacement times are negligible.

(ii) Suppose that the failure times of each unit are independent, and have identical density $f(t)$ and distribution $F(t)$. Then, the failure rate or the hazard rate is $r(t) \equiv f(t)/[1 - F(t)]$ and its cumulative hazard rate is $H(t) \equiv \int_0^t r(u)\, du$, *i.e.*, $\overline{F}(t) \equiv 1 - F(t) = e^{-H(t)}$.

(iii) A cost c_m is the cost of minimal repair, and c_p is the cost of scheduled replacement.

Then, the expected cost of one interval $(0, T]$ is, from Barlow and Proschan (1965),

$$\tilde{C}(1) \equiv c_m H(T) + c_p = c_m H\left(\frac{S}{n}\right) + c_p. \qquad (4.1)$$

Thus, the total expected cost until time S is

$$C(n) \equiv n\tilde{C}(1) = n\left[c_m H\left(\frac{S}{n}\right) + c_p\right] \qquad (n = 1, 2, \ldots). \qquad (4.2)$$

We find an optimal partition number n^* which minimizes $C(n)$ in (4.2). Evidently,

$$C(1) = c_m H(S) + c_p,$$

$$C(\infty) \equiv \lim_{n \to \infty} C(n) = \infty.$$

Hence, there exists a finite number n^* $(1 \leq n^* < \infty)$ which minimizes $C(n)$. Forming the inequality $C(n+1) - C(n) \geq 0$, we have

$$\frac{1}{nH\left(\dfrac{S}{n}\right) - (n+1)H\left(\dfrac{S}{n+1}\right)} \geq \frac{c_m}{c_p} \qquad (n = 1, 2, \ldots). \tag{4.3}$$

When the failure time has a Weibull distribution, *i.e.*, $H(t) = \lambda t^m$ $(m > 1)$, equation (4.3) becomes

$$\frac{1}{\dfrac{1}{n^{m-1}} - \dfrac{1}{(n+1)^{m-1}}} \geq \frac{\lambda c_m}{c_p} \cdot S^m \qquad (n = 1, 2, \ldots). \tag{4.4}$$

Since it is easily proved that $[1/x]^\alpha - [1/(x+1)]^\alpha$ is strictly decreasing in x for $1 \leq x < \infty$ and $\alpha > 0$, the left-hand side of (4.4) is strictly increasing in n to ∞. Thus, there exists a finite and unique minimum n^* $(1 \leq n^* < \infty)$ which satisfies (4.4). In particular, when $m = 2$, *i.e.*, $H(t) = \lambda t^2$, equation (4.4) is

$$\frac{n(n+1)}{2} \geq \frac{\lambda c_m}{2c_p} \cdot S^2 \qquad (n = 1, 2, \ldots), \tag{4.5}$$

which agrees with the type of inequality (4.3) in Nakagawa *et al.* (2004).

To obtain an optimal n^*, putting that $T = S/n$, equation (4.2) is

$$C(T) = S\left[\frac{c_m H(T) + c_p}{T}\right]. \tag{4.6}$$

Thus, the problem which minimizes $C(T)$ corresponds to the problem of the standard replacement with minimal repair for an infinite interval. Many discussions on such optimal policies have been made [Nakagawa (1979, 1981), Valdez-Flores and Feldman (1989)]: Differentiating $C(T)$ with respect to T and setting it equal to zero, we have

$$\int_0^T [r(T) - r(t)]\, dt = \frac{c_p}{c_m}. \tag{4.7}$$

When the failure rate is strictly increasing, a solution $\tilde{T}$ to (4.7) is unique if it exists.

Therefore, we have the following optimal policy [Nakagawa *et al.* (2004)]:

(i) If $\tilde{T} < S$ then we put that $[S/\tilde{T}] \equiv n$ and calculate $C(n)$ and $C(n+1)$ from (4.2), where $[x]$ denotes the greatest integer in x. If $C(n) \leq C(n+1)$ then $n^* = n$, and conversely, if $C(n) > C(n+1)$ then $n^* = n+1$.

(ii) If $\tilde{T} \geq S$ then $n^* = 1$.

4.2.2 Block replacement

Suppose that a unit is always replaced at any failure between replacements. This is called block replacement and has been studied by many authors [Nakagawa (1989), and Gertsbakh (2003)].

We define the block replacement model for a finite interval.

(i) The unit is replaced at periodic times kT $(k = 1, 2, \dots)$ or any failure between replacements. After that, any unit is as good as new by each replacement. It is assumed that the replacement times are negligible.

(ii) Let $M(t)$ be the renewal function of distribution $F(t)$, *i.e.*, $M(t) \equiv \sum_{n=1}^{\infty} F^{(n)}(t)$, where $F^{(n)}(t)$ is the n-th Stieltjes convolution of $F(t)$, and $F^{(n)}(t) \equiv \int_0^t F^{(n-1)}(t-u)\,dF(u)$ $(n = 1, 2, \dots)$ and $F^{(0)}(t) \equiv 1$ for $t \geq 0$, 0 for $t < 0$. That is, $M(t)$ represents the expected number of failed units during $(0, t]$.

(iii) A cost c_f is the cost of replacement for a failed unit, and c_p is the cost of scheduled replacement.

Then, the expected cost of one interval $(0, T]$ is, from Barlow and Proschan (1965) ,

$$\tilde{C}(1) \equiv c_f M(T) + c_p = c_f M\left(\frac{S}{n}\right) + c_p. \tag{4.8}$$

Thus, the total expected cost until time S is

$$C(n) \equiv n\tilde{C}(1) = n\left[c_f M\left(\frac{S}{n}\right) + c_p\right] \qquad (n = 1, 2, \dots). \tag{4.9}$$

From the inequality $C(n+1) - C(n) \geq 0$, we have

$$\frac{1}{nM\left(\frac{S}{n}\right) - (n+1)M\left(\frac{S}{n+1}\right)} \geq \frac{c_f}{c_p} \qquad (n = 1, 2, \dots), \tag{4.10}$$

and putting that $T = S/n$ in (4.9),

$$C(T) = S\left[\frac{c_f M(T) + c_p}{T}\right], \tag{4.11}$$

which corresponds to the standard block replacement. Let $m(t)$ be a renewal density of F, *i.e.*, $m(t) \equiv M'(t)$. Then, differentiating $C(T)$ with respect to T and setting it equal to zero, we have

$$\int_0^T [m(T) - m(t)]\, dt = \frac{c_p}{c_f}. \tag{4.12}$$

Therefore, by obtaining $\tilde{T}$ which satisfies (4.12) and applying it to the optimal policy, we can get an optimal replacement number n^* which minimizes $C(n)$ in (4.9).

4.2.3 Simple replacement

Suppose that failures of a unit are replaced only at times kT $(k = 1, 2, \dots, n)$, which is called *Policy II* in Nakagawa (1979) . This model is defined as follows:

(i) The unit is replaced only at periodic times kT $(k = 1, 2, \dots n)$, where $n \equiv S/T$. If the unit fails before the next replacement, its failure remains until the next replacement.

(ii) The failure time has a general distribution $F(t)$ with finite mean $1/\lambda$, where $\overline{F}(t) \equiv 1 - F(t)$.

(iii) A cost c_d is the cost per unit of time for the time elapsed between a failure and its detection, and c_p is the cost of scheduled replacement.

Then, the expected cost of one interval $(0, T]$ is, from Nakagawa (1979) ,

$$\tilde{C}(1) \equiv c_d \int_0^T F(t)\, dt + c_p = c_d \int_0^{S/n} F(t)\, dt + c_p. \tag{4.13}$$

Thus, the total expected cost until time S is

$$C(n) \equiv n\tilde{C}(1) = n \left[c_d \int_0^{S/n} F(t)\, dt + c_p \right] \qquad (n = 1, 2, \dots). \tag{4.14}$$

Evidently, since

$$C(1) = c_d \int_0^S F(t)\, dt + c_p,$$

$$C(\infty) \equiv \lim_{n\to\infty} C(n) = \infty,$$

there exists a finite n^* $(1 \le n^* < \infty)$ which minimizes $C(n)$. Forming the inequality $C(n+1) - C(n) \ge 0$ implies

$$\frac{1}{n \int_0^{S/n} F(t)\, dt - (n+1) \int_0^{S/(n+1)} F(t)\, dt} \ge \frac{c_d}{c_p} \qquad (n = 1, 2 \dots). \tag{4.15}$$

In particular, when $F(t) = 1 - e^{-\lambda t}$, equation (4.15) is

$$\frac{1}{n e^{-\lambda S/n} - (n+1) e^{-\lambda S/(n+1)}} \ge \frac{\lambda c_d}{c_p} \qquad (n = 1, 2, \dots). \tag{4.16}$$

Using the approximation of $e^{-a} \approx 1 - a + a^2/2$ for small a, equation (4.16) is rewritten as

$$\frac{n(n+1)}{2} \ge \frac{\lambda c_d}{c_p + \lambda c_d} \cdot \left(\frac{\lambda S}{2} \right)^2 \qquad (n = 1, 2, \dots), \tag{4.17}$$

which agrees with the type of inequality (4.3) in Nakagawa *et al.* (2004).

Putting that $T = S/n$, equation (4.14) is

$$C(T) = S \left[\frac{c_d \int_0^T F(t)\, dt + c_p}{T} \right]. \tag{4.18}$$

Differentiating $C(T)$ with respect to T and setting it equal to zero, we have

$$\int_0^T [F(T) - F(t)]\, dt = \frac{c_p}{c_d}. \tag{4.19}$$

Noting that the left-hand side of (4.19) is strictly increasing from 0 to $1/\lambda$, there exists a finite and unique $\tilde{T}$ which satisfies (4.19), if $1/\lambda > c_p/c_d$. Therefore, using the optimal policy, we can get an optimal replacement number n^* which minimizes $C(n)$ in (4.14).

In general, the above results of three replacements are summarized as follows: The total expected cost until time S is

$$C(n) = n\left[c_j\Phi\left(\frac{S}{n}\right) + c_p\right] \qquad (n = 1, 2, \dots), \tag{4.20}$$

where $\Phi(t)$ is $H(t)$, $M(t)$ and $\int_0^t F(u)\, du$, and c_j is c_m, c_f and c_d for the respective periodic, block and simple replacements. Forming the inequality $C(n+1) - C(n) \geq 0$ yields

$$\frac{1}{n\Phi\left(\frac{S}{n}\right) - (n+1)\Phi\left(\frac{S}{n+1}\right)} \geq \frac{c_j}{c_p} \qquad (n = 1, 2, \dots). \tag{4.21}$$

Putting that $T = S/n$ in (4.20), we have

$$C(T) = S\left[\frac{c_j\Phi(T) + c_p}{T}\right], \tag{4.22}$$

and differentiating $C(T)$ with respect to T and setting it equal to zero,

$$T\Phi'(T) - \Phi(T) = \frac{c_p}{c_j}. \tag{4.23}$$

If there exists a solution $\tilde{T}$ to (4.23), then we can get an optimal number n^* for each replacement, using the optimal policy.

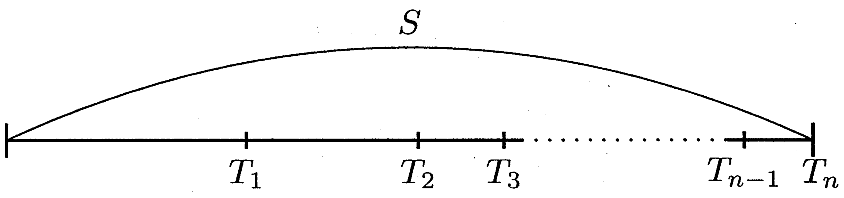

Figure 4.2: Finite interval S with n sequential intervals.

4.3 Inspection Policy

Nakagawa *et al.* (2004) have considered the periodic inspection model for a finite interval $(0, S]$. In this section, we extend this model to a sequential inspection policy as follows:

(i) An operating unit is checked at successive times $0 < T_1 < T_2 < \cdots < T_n$ (see Figure 4.2), where $T_0 \equiv 0$ and $T_n \equiv S$.

(ii) The failure time has a general distribution $F(t)$ with finite mean $1/\lambda$, where $\overline{F}(t) \equiv 1 - F(t)$.

(iii) A cost c_i is the cost of one check and c_d is the cost per unit of time for the time elapsed between a failure and its detection at the next check.

Then, the total expected cost until the detection of failure or time S is

$$\mathbf{C}(n) = \sum_{k=0}^{n-1} \int_{T_k}^{T_{k+1}} [c_i(k+1) + c_d(T_{k+1} - t)]dF(t) + c_i n\overline{F}(S) \qquad (n = 1, 2, \dots). \tag{4.24}$$

Putting that $\partial\mathbf{C}/\partial T_k = 0$, we have

$$T_{k+1} - T_k = \frac{F(T_k) - F(T_{k-1})}{f(T_k)} - \frac{c_i}{c_d} \qquad (k = 1, 2, \dots, n-1), \tag{4.25}$$

and the resulting expected cost is

$$\mathbf{C}(n) + c_d \int_0^S \bar{F}(t)\, dt = \sum_{k=0}^{n-1} [c_i + c_d(T_{k+1} - T_k)]\bar{F}(T_k) \qquad (n = 1, 2, \dots). \tag{4.26}$$

For example, when $n = 3$, the checking times T_1 and T_2 are given by the solutions of equations

$$S - T_2 = \frac{F(T_2) - F(T_1)}{f(T_2)} - \frac{c_i}{c_d},$$

$$T_2 - T_1 = \frac{F(T_1)}{f(T_1)} - \frac{c_i}{c_d},$$

and the total expected cost is

$$\mathbf{C}(3) + c_d \int_0^S \bar{F}(t)\,dt = c_i + c_d T_1 + [c_i + c_d(T_2 - T_1)]\bar{F}(T_1) + [c_i + c_d(S - T_2)]\bar{F}(T_2).$$

Therefore, we compute optimal T_k $(k = 1, 2, \ldots, n-1)$ which satisfies (4.25), and substituting them into (4.26), we obtain the total expected cost $\mathbf{C}(n)$. Next, comparing $\mathbf{C}(n)$ for all $n \geq 1$, we can get an optimal checking number n^* and checking times T_k^* $(k = 1, 2, \ldots, n^*)$.

4.4 Numerical Examples

We compute numerically optimal policies for each model. Table 4.1 shows optimal n^* for periodic replacement with minimal repair for $1/\lambda = 10$, 20, 30, 40, 50, 60, 70, 80, 90, 100, and $c_p = 5$, 6, 7, 8, 9, 10, 15, 20, 25 when $S = 100$, $c_m = 1$ and $F(t) = 1 - e^{-\lambda t^2}$. This indicates that n^* decreases as $1/\lambda$ or c_p increases.

Table 4.2 shows optimal n^* for block replacement for $1/\lambda = 10$, 20, 30, 40, 50, 60, 70, 80, 90, 100, and $c_p = 0.04$, 0.06, 0.08, 0.1, 0.12, 0.14, 0.16, 0.18, 0.2 when $S = 100$, $c_f = 1$ and $F(t)$ is a gamma distribution with parameter 2, *i.e.*,

$$m(t) = \frac{\lambda}{2} - \frac{\lambda}{2} e^{-2\lambda t}, \qquad M(t) = \frac{\lambda t}{2} - \frac{1}{4} - \frac{1}{4} e^{-2\lambda t}.$$

Table 4.3 shows optimal n^* for simple replacement for $1/\lambda = 10$, 20, 30, 40, 50, 60, 70, 80, 90, 100, and $c_p = 0.5$, 0.6, 0.7, 0.8, 0.9, 1, 2, 4, 6 when $S = 100$, $c_d = 1$ and

Table 4.1: Optimal n^* for periodic replacement with minimal repair when $S = 100$, $c_m = 1$ and $F(t) = 1 - e^{-\lambda t^2}$.

	c_p								
$1/\lambda$	5	6	7	8	9	10	15	20	25
10	14	13	12	11	11	10	8	7	6
20	10	9	8	8	7	7	6	5	4
30	8	7	7	6	6	6	5	4	4
40	7	6	6	6	5	5	4	4	3
50	6	6	5	5	5	4	4	3	3
60	6	5	5	5	4	4	3	3	3
70	5	5	5	4	4	4	3	3	2
80	5	5	4	4	4	4	3	3	2
90	5	4	4	4	4	3	3	2	2
100	4	4	4	4	3	3	3	2	2

Table 4.2: Optimal n^* for block replacement when $S = 100$, $c_f = 1$ and $F(t)$ is gamma with parameter 2.

	c_p								
$1/\lambda$	0.04	0.06	0.08	0.1	0.12	0.14	0.16	0.18	0.2
10	28	21	17	15	12	11	9	8	7
20	14	11	9	7	6	5	5	4	3
30	9	7	6	5	4	4	3	3	2
40	7	5	4	4	3	3	2	2	2
50	5	4	3	3	2	2	2	2	1
60	4	4	3	2	2	2	2	1	1
70	4	3	3	2	2	2	1	1	0
80	4	3	2	2	2	1	1	0	0
90	3	2	2	2	1	1	1	0	0
100	3	2	2	1	1	1	0	0	0

Table 4.3: Optimal n^* for simple replacement when $S = 100$, $c_d = 1$ and $F(t) = 1-e^{-\lambda t}$.

	c_p								
$1/\lambda$	0.5	0.6	0.7	0.8	0.9	1	2	4	6
10	28	25	23	21	20	19	12	7	5
20	21	19	17	16	15	14	9	6	5
30	17	16	14	13	12	12	8	5	4
40	15	14	13	12	11	10	7	5	4
50	13	12	11	11	10	9	6	4	3
60	12	11	10	10	9	9	6	4	3
70	11	10	10	9	8	8	6	4	3
80	11	10	9	8	8	7	5	4	3
90	10	9	9	8	7	7	5	3	3
100	10	9	8	8	7	7	5	3	3

Table 4.4: Checking times T_k and expected cost $\tilde{\mathbf{C}}(n) \equiv \mathbf{C}(n)/c_d + \int_0^S \overline{F}(t)dt$ when $S = 100$, $c_i/c_d = 2$ and $F(t) = 1 - e^{-\lambda t^2}$.

n	1	2	3	4	5	6	7	8	9
T_1	100	64,1	50.9	44.1	40.3	38.1	36.8	36.3	36.1
T_2		100	77.1	66.0	50.8	56.2	54.3	53.3	53.1
T_3			100	84.0	75.4	70.5	67.8	66.6	66.3
T_4				100	88.6	82.3	78.9	77.3	77.0
T_5					100	92.1	87.9	85.9	85.5
T_6						100	94.9	92.5	92.0
T_7							100	97.2	96.6
T_8								100	99.3
T_9									100
$\tilde{\mathbf{C}}(n)$	10.20	93.44	91.52	91.16	91.47	92.11	92.91	93.79	94.70

$F(t) = 1 - e^{-\lambda t}$. This shows similar results with Tables 4.1 and 4.2, *i.e.*, n^* decreases as $1/\lambda$ or c_p increases.

Table 4.4 gives the checking time T_k $(k = 1, 2, \ldots, n)$ and the expected cost $\tilde{\mathrm{C}}(n) \equiv \mathrm{C}(n)/c_d + \int_0^S \bar{F}(t)\,dt$ when $S = 100$, $c_i/c_d = 2$ and $F(t) = 1 - e^{-\lambda t^2}$. In this case, we set that the mean failure time is equal to S, *i.e.*,

$$\int_0^\infty e^{-\lambda t^2}\,dt = \frac{1}{2}\sqrt{\frac{\pi}{\lambda}} = S.$$

Comparing $\tilde{\mathrm{C}}(n)$ for $n = 1, 2, \ldots, 9$, the expected cost is minimum at $n = 4$. That is, the optimal checking number is $n^* = 4$ and checking times are 44.1, 66.0, 84.0, 100.

4.5 Conclusions

We have derived optimal policies for periodic, block and simple replacement for a finite interval, using the known results of standard replacements and the partition method. Further, we have shown the computing method of obtaining optimal sequential times of an inspection model. Such computations might be more troublesome than those of an infinite case. But, it would be easy to compute optimal times numerically even by personal computers, as they have greatly developed.

In this chapter, we have made no mention of age replacement. We can obtain an optimal age replacement policy for a finite interval by the similar method as follows: We divide a whole working time S into n equal parts, *i.e.*, $nT \equiv S$, and derive an optimal replacement time for one interval $(0, T]$. Further, by the partition method, we determine an optimal replacement number n^*. If a unit is replaced at time T_0 $(0 < T_0 \leq T)$ then we may reconsider the same replacement policy for the remaining interval $S - T_0$.

Chapter 5

Optimal Policies for a System with Two Types of Inspection

This chapter considers optimal inspection policies for a system with two types of inspection: Type-1 inspection is done so frequently more than type-2 inspection, because the loss cost for one check of type-1 inspection is lower than that of type-2 inspection. On the other hand, there exist some failures which can not be detected by type-1 inspection and can be detected only by type-2 inspection. Optimal inspection policies for such a system are considered. Optimal numbers which minimize the expected costs are analytically derived. Numerical examples are given when the failure time distribution is exponential.

5.1 Introduction

This chapter considers a system which is checked periodically by type-1 inspection or type-2 inspection. Suppose that the cost of type-1 inspection is lower than that of type-2 inspection. Therefore, type-1 inspection checks the system more frequently than type-2 inspection. On the other hand, it is assumed that type-2 inspection can detect any failure which can not be detected by type-1 inspection.

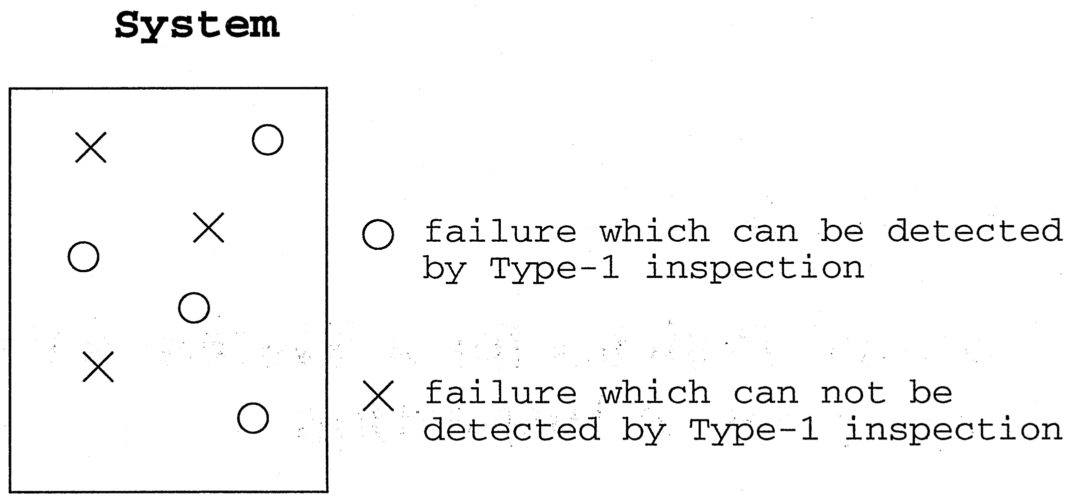

Figure 5.1: System with two types of inspection.

A typical example of such a inspection policy in real systems is electronic control devices which are periodically checked by self-diagnosis. The self-diagnosis function of the system is embedded in electric circuits, and check it periodically [O'Connor (2001), Jha and Gupta (2003)]. On the other hand, the system complexity has dramatically increased, and as a result, it has been difficult to design the self-diagnosis which can detect all possible failures. Moreover, the cost performance of self-diagnosis increases as the coverage to detect failures increases. The external inspection with tester has a complex implement and its cost is high. Therefore, the inspection should be classified into two cases that the high-cost inspection and low-cost self-diagnosis, where intervals of high-cost inspection would be larger than those of low-cost self-diagnosis. Two types of inspection policies for storage systems were studied by Kodo, Nakagawa and Nishi (1995).

The inspection policy in reliability theory is applied to such a model: Type-1 inspection checks a system at periodic times jT $(j = 1, 2, \dots)$, and the type-2 inspection checks a system at periodic time kmT $(k = 1, 2, \dots)$. When the system fails, its failure is classified with a probability, *i.e.*, the failure can be detected by type-1 inspection with probability p. On the other hand, some failures can not be detected by type-1 inspection with probability $1 - p$ (see Figure 5.1).

Consider the time from the beginning of system operation to the detection of failure as one cycle, and further, introduce a loss cost for the time elapsed between a failure and its detection. Then, the mean time and the total expected cost of one cycle, and the expected cost per unit of time are derived. Optimal numbers m^* which minimize the expected costs are analytically derived. Finally, numerical examples are given when the failure time distribution is exponential.

5.2 Model and Assumptions

Consider a system which is checked periodically by two-types of inspection: Type-1 inspection is done so frequently more than type-2 inspection, because the loss cost for one check of type-1 inspection is lower than that of type-2 inspection. Whereas, there exist some failures which can not be detected by type-1 inspection.

For this model, we define the following assumptions:

(i) The system is checked by two types of inspection; type-1 or type-2 inspection. The system is replaced when its failure is detected by inspection. Any failure does not occur between the first failure and the next inspection. If the failure is detected then the system is maintained and is as good as new.

(ii) The system is checked periodically by two types of inspection: Type-1 inspection is performed at periodic times jT ($j = 1, 2, \dots$) and type-2 inspection is performed at periodic times kmT ($k = 1, 2, \dots$) for some specified T and m ($m = 1, 2, \dots$), *i.e.*, type-2 inspection is done at every m times of type-1 inspection.

(iii) The failure time distribution has a general distribution $F(t)$ with finite mean $1/\lambda$, where $\overline{F}(t) \equiv 1 - F(t)$.

(iv) When the system fails, its failure is classified in the following way: The failure can be detected by type-1 inspection and type-2 inspection with probability p $(0 < p \leq 1)$. On the other hand, the failure can not be detected by type-1 inspection with probability $1 - p$, and can be detected only by type-2 inspection. In other words, type-2 inspection can detect any failure.

(v) Let c_{i1} be the cost of one check by type-1 inspection, c_{i2} be the cost of one check by type-2 inspection, that is, the inspection cost at time kmT includes two costs of the type-1 inspection and type-2 inspection. Further, let c_d be the loss cost per unit of time for the time elapsed between a failure and its detection.

Figure 5.2 shows the processes of the system with two types of inspection: The horizontal axis represents the process of time. Upper side shows that when the system fails at time t $(kmT + jT < t \leq kmT + (j+1)T)$, its failure is detected by type-1 inspection at time $kmT + (j+1)T$ with probability p, and the lower side shows that the failure is detected only by type-2 inspection at time $(k+1)mT$ with probability $1 - p$.

We define one cycle as the time from the beginning of system operation to the detection of failure. Then, the mean time of one cycle is given by

$$\begin{aligned} A(m;T) &= p\sum_{k=0}^{\infty}\sum_{j=0}^{m-1}\int_{kmT+jT}^{kmT+(j+1)T}[kmT + (j+1)T]\,dF(t) + (1-p)\sum_{k=0}^{\infty}\int_{kmT}^{(k+1)mT}(k+1)mT\,dF(t) \\ &= pT\sum_{k=0}^{\infty}\overline{F}(kT) + (1-p)mT\sum_{k=0}^{\infty}\overline{F}(kmT) \qquad (m = 1, 2, \dots). \end{aligned} \tag{5.1}$$

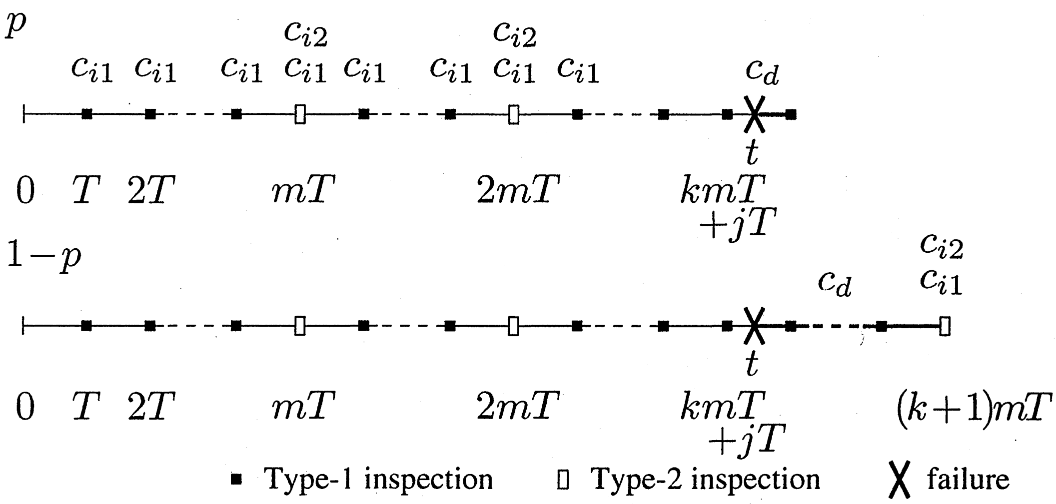

Figure 5.2: Two types of inspection.

Further, the total expected cost of one cycle is

$$B(m;T) = p\sum_{k=0}^{\infty}\sum_{j=0}^{m-1}\int_{kmT+jT}^{kmT+(j+1)T}\{c_{i1}(km+j+1) + c_{i2}k+c_d[kmT+(j+1)T-t]\}\,dF(t)$$

$$+ (1-p)\sum_{k=0}^{\infty}\int_{kmT}^{(k+1)mT}\{(c_{i1}m + c_{i2})(k+1) + c_d[(k+1)mT - t]\}\,dF(t)$$

$$= (c_{i1}+c_dT)\left[p\sum_{k=0}^{\infty}\overline{F}(kT) + (1-p)m\sum_{k=0}^{\infty}\overline{F}(kmT)\right] + c_{i2}\sum_{k=0}^{\infty}\overline{F}(kmT) - pc_{i2} - \frac{c_d}{\lambda}. \tag{5.2}$$

Thus, the expected cost $C(m;T)$ per unit of time is, from (5.1) and (5.2),

$$C(m;T) \equiv \frac{B(m;T)}{A(m;T)}$$

$$= \frac{c_{i1}\left[p\sum_{k=0}^{\infty}\overline{F}(kT)+(1-p)m\sum_{k=0}^{\infty}\overline{F}(kmT)\right] + c_{i2}\left[\sum_{k=0}^{\infty}\overline{F}(kmT) - p\right] - \frac{c_d}{\lambda}}{\left[p\sum_{k=0}^{\infty}\overline{F}(kT) + (1-p)m\sum_{k=0}^{\infty}\overline{F}(kmT)\right]T}$$

$$+ c_d \qquad (m = 1,2,\dots). \tag{5.3}$$

5.3 Optimal Policy 1

Assume that the failure distribution is exponential, *i.e.*, $F(t) = 1 - e^{-\lambda t}$. Then, the total expected cost $B(m;T)$ in (5.2) can be rewritten as

$$B(m;T) = (c_{i1}+c_dT)\left[\frac{p}{1-e^{-\lambda T}} + \frac{(1-p)m}{1-e^{-\lambda mT}}\right] + \frac{c_{i2}}{1-e^{-\lambda mT}} - pc_{i2} - \frac{c_d}{\lambda} \quad (m = 1, 2, \dots), \tag{5.4}$$

and the expected cost $C(m;T)$ in (5.3) is

$$C(m;T) = c_d + \frac{c_{i1}}{T} + \frac{c_{i2} - \left(\frac{1}{\lambda}c_d + pc_{i2}\right)(1 - e^{-\lambda mT})}{(1-p)m(1 - e^{-\lambda T}) + p(1 - e^{-\lambda mT})}\left(\frac{1 - e^{-\lambda T}}{T}\right) \quad (m = 1, 2, \dots). \tag{5.5}$$

We seek an optimal number m_1^* of type-2 inspection which minimizes the total expected cost $B(m;T)$ in (5.4) for a fixed $T > 0$. Letting $B(m+1;T) \geq B(m;T)$, we have

$$\sum_{k=1}^{m}(e^{\lambda kT} - 1) \geq \frac{c_{i2}}{(1-p)(c_{i1} + c_dT)}. \tag{5.6}$$

It is easily seen that the left-hand side of (5.6) is strictly increasing in m from $e^{\lambda T} - 1$ to ∞. Thus, there exists a finite and unique minimum m_1^* $(1 \leq m_1^* < \infty)$ which satisfies (5.6).

In particular, since $e^{\lambda kT} - 1 > \lambda kT$, if there exists a minimum solution $\overline{m}_1$ to satisfy the following inequality:

$$\sum_{k=1}^{n} k = \frac{m(m+1)}{2} \geq \frac{c_{i2}}{\lambda T(1-p)(c_{i1} + c_dT)}, \tag{5.7}$$

then $m_1^* \leq \overline{m}_1$. It is further noted from (5.6) that optimal m_1^* is decreasing with both of $1-p$ and T, and $m_1^* \to \infty$ as $p \to 1$.

5.4 Optimal Policy 2

It is assumed that $c_d/\lambda > c_{i2}$, *i.e.*, the downtime cost for the mean failure time is greater than the additional cost for one check of type-2 inspection. Then, we seek an optimal number m_2^* which minimizes the total expected cost $C(m;T)$ in (5.5). Letting $C(m+1;T) \geq C(m;T)$, we have

$$\frac{\sum_{k=1}^{m}(e^{\lambda kT}-1)}{m(1-p)+\dfrac{1}{1-e^{-\lambda T}}} \geq \frac{c_{i2}}{(1-p)\left[\dfrac{c_d}{\lambda}-(1-p)c_{i2}\right]}. \tag{5.8}$$

Letting denote the left-side hand of (5.8) by $L(m)$,

$$L(1) = \frac{e^{\lambda T}-1}{1-p+\dfrac{1}{1-e^{-\lambda T}}},$$

$$L(\infty) = \infty,$$

$$L(m+1)-L(m) = \frac{(1-p)\sum_{k=1}^{m}(e^{\lambda(m+1)T}-e^{\lambda kT})+\dfrac{e^{\lambda(m+1)T}-1}{1-e^{-\lambda T}}}{\left[m(1-p)+\dfrac{1}{1-e^{-\lambda T}}\right]\left[(m+1)(1-p)+\dfrac{1}{1-e^{-\lambda T}}\right]} > 0.$$

Thus, $L(m)$ is strictly increasing from $L(1)$ to ∞, and hence, there exists a finite and unique minimum m_2^* $(1 \leq m_2^* < \infty)$ which satisfies (5.8).

Since $e^{\lambda kT}-1 > \lambda kT$, if there exists a minimum solution to satisfy the following inequality:

$$\frac{\sum_{k=1}^{m} k}{m(1-p)+\dfrac{1}{\lambda T}} \geq \frac{c_{i2}}{\lambda T(1-p)\left[\dfrac{c_d}{\lambda}-(1-p)c_{i2}\right]}, \tag{5.9}$$

then $m_2^* \leq \overline{m}_2$. It is further noted that optimal m_2^* has no relation with c_{i1}, and is decreasing with both of $1-p$ and T, and $m_2^* \to \infty$ as $p \to 1$.

5.5 Numerical Examples

We compute numerically optimal inspection numbers m_1^* and m_2^* which minimize the expected costs $B(m;T)$ and $C(m;T)$ when $F(t) = 1 - e^{-\lambda t}$, respectively, and compare m_1^* with $\overline{m}_1$ and m_2^* with $\overline{m}_2$. All costs are normalized to c_{i1} as a unit cost, *i.e.*, they are divided by c_{i1}.

Table 5.1 presents the optimal number m_1^* which minimizes $B(m;T)$ and its upper bound $\overline{m}_1$ for $1/(\lambda T) = 300$, 600, $c_dT/c_{i1} = 100$, 1000 and $c_{i2}/c_{i1} = 1$, 2, 5, 10, 15, 20, 25, 30 when $p = 0.9$. This indicates that m_1^* increases as c_{i2}/c_{i1} or $1/(\lambda T)$ increases and c_dT/c_{i1} decreases. For example, when the interval of type-1 inspection is $T = 1$ day, $1/\lambda = 300$, $c_d/c_{i1} = 100$ and $p = 0.9$, type-2 inspection should be performed almost every month for $c_{i2}/c_{i1} = 15$.

Table 5.2 shows the optimal number m_1^* which minimizes $B(m;T)$ and $\overline{m}_1$ for $1/(\lambda T) = 300, 600$, $c_dT/c_{i1} = 100, 1000$ and $p = 0.5, 0.7, 0.8, 0.9, 1.0$ when $c_{i2}/c_{i1} = 10$. This indicates that the optimal m_1^* decreases with $1-p$. Thus, if $1-p$ is large, it would be better to perform type-2 inspection early. When $p = 1.0$, type-2 inspection should not be performed, because all failures are detected by type-1 inspection with low-cost.

Table 5.3 gives the optimal number m_2^* which minimizes $C(m;T)$ and its upper bound $\overline{m}_2$ for $1/(\lambda T) = 300$, 600, $c_dT/c_{i1} = 100$, 1000 and $c_{i2}/c_{i1} = 1$, 2, 5, 10, 15, 20, 25, 30 when $p = 0.9$. This indicates that m_2^* is a little larger than m_1^* in Table 5.1.

Table 5.4 presents the optimal number m_2^* which minimizes $C(m;T)$ and $\overline{m}_2$ for $1/(\lambda T) = 300$, 600, $c_dT/c_{i1} = 100$, 1000 and $p = 0.5$, 0.6, 0.7, 0.8, 0.9, 1.0 when $c_{i2}/c_{i1} = 10$. For example, when the interval of type-1 inspection is $T = 1$ day, $1/\lambda = 600$ and $c_d/c_{i1} = 100$, type-2 inspection should be performed almost every 34 days for $p = 0.9$ and every 16 days for $p = 0.5$.

It is of interest that the upper bounds $\overline{m}_i$ $(i = 1, 2)$ give close approximations to

Table 5.1: Optimal number m_1^* to minimize $B(m;T)$ for $1/(\lambda T)$, c_{i2}/c_{i1} and c_dT/c_{i1} when $p = 0.9$.

$\frac{c_{i2}}{c_{i1}}$	$1/(\lambda T) = 300$				$1/(\lambda T) = 600$			
	c_dT/c_{i1}							
	100		1000		100		1000	
	m_1^*	$\overline{m}_1$	m_1^*	$\overline{m}_1$	m_1^*	$\overline{m}_1$	m_1^*	$\overline{m}_1$
1	8	8	2	2	11	11	3	3
2	11	11	3	3	15	15	5	5
5	17	17	5	5	24	24	8	8
10	24	24	8	8	34	34	11	11
15	29	30	9	9	42	42	13	13
20	34	34	11	11	48	49	15	15
25	38	39	12	12	54	55	17	17
30	41	42	13	13	59	60	19	19

Table 5.2: Optimal number m_1^* to minimize $B(m;T)$ for $1/(\lambda T)$, p and c_dT/c_{i1} when $c_{i2}/c_{i1} = 10$.

p	$1/(\lambda T) = 300$				$1/(\lambda T) = 600$			
	c_dT/c_{i1}							
	100		1000		100		1000	
	m_1^*	$\overline{m}_1$	m_1^*	$\overline{m}_1$	m_1^*	$\overline{m}_1$	m_1^*	$\overline{m}_1$
0.5	11	11	3	3	15	15	5	5
0.6	12	12	4	4	17	17	5	5
0.7	14	14	4	4	20	20	6	6
0.8	17	17	5	5	24	24	8	8
0.9	24	24	8	8	34	34	11	11
1.0	∞	∞	∞	∞	∞	∞	∞	∞

Table 5.3: Optimal number m_2^* to minimize $C(m;T)$ for $1/(\lambda T)$, c_{i2}/c_{i1} and c_dT/c_{i1} when $p = 0.9$.

c_{i2}/c_{i1}	$1/(\lambda T) = 300$				$1/(\lambda T) = 600$			
	c_dT/c_{i1}							
	100		1000		100		1000	
	m_2^*	$\overline{m}_2$	m_2^*	$\overline{m}_2$	m_2^*	$\overline{m}_2$	m_2^*	$\overline{m}_2$
1	8	8	2	3	11	11	3	4
2	11	11	3	4	15	16	5	5
5	17	17	5	6	24	25	8	8
10	24	24	8	8	34	35	11	11
15	30	30	9	10	42	43	13	13
20	34	35	11	11	49	49	15	16
25	38	39	12	12	54	55	17	17
30	42	43	13	13	59	60	19	19

Table 5.4: Optimal number m_2^* to minimize $C(m;T)$ for $1/(\lambda T)$, p, and c_dT/c_{i1} when $c_{i2}/c_{i1} = 10$.

p	$1/(\lambda T) = 300$				$1/(\lambda T) = 600$			
	c_dT/c_{i1}							
	100		1000		100		1000	
	m_2^*	$\overline{m}_2$	m_2^*	$\overline{m}_2$	m_2^*	$\overline{m}_2$	m_2^*	$\overline{m}_2$
0.5	11	11	4	4	16	16	5	5
0.6	12	12	4	4	17	17	6	6
0.7	14	14	5	5	20	20	6	6
0.8	17	17	5	6	24	25	8	8
0.9	24	25	8	8	34	35	11	11
1.0	∞	∞	∞	∞	∞	∞	∞	∞

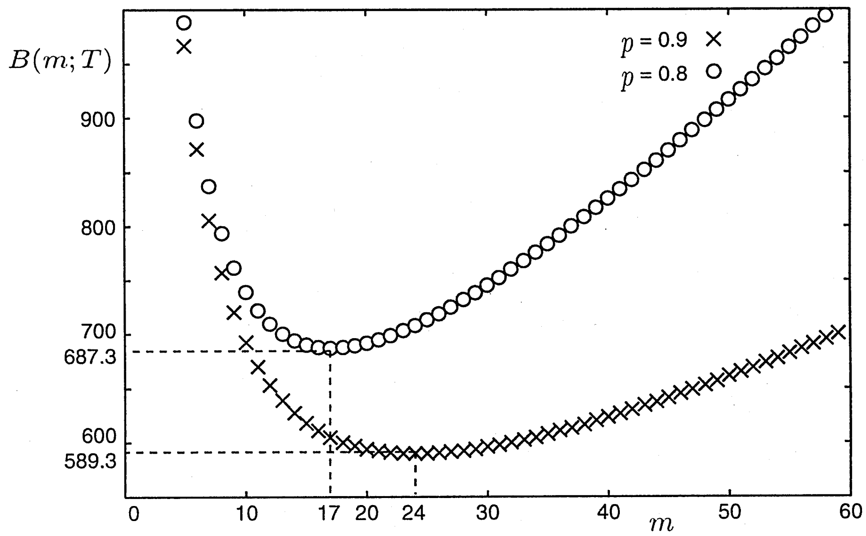

Figure 5.3: Expected cost $B(m;T)$.

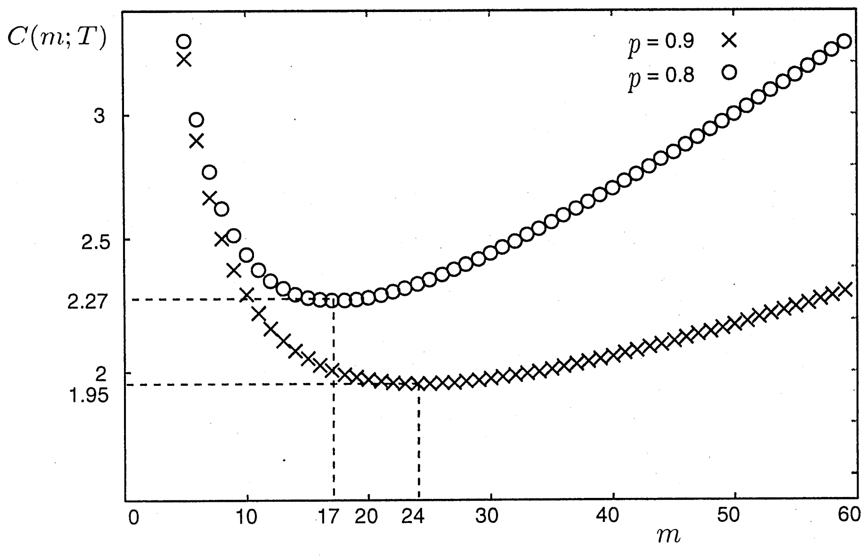

Figure 5.4: Expected cost $C(m;T)$.

optimal numbers in all tables.

Figure 5.3 draws the total expected cost $B(m;T)$ for $p = 0.8, 0.9$ when $1/(\lambda T) = 300$, $c_{i2}/c_{i1} = 10$, $c_d T/c_{i1} = 100$. For example, when $p = 0.9$, the optimal number is $m^* = 24$ and $B(m^*;T) = 589.3$. This indicates that $B(m;T)$ decreases with p. Thus, to decrease the expected cost, we have to increase the probability p of detecting failures by type-1 inspection.

Figure 5.4 draws the expected cost $C(m;T)$ for $p = 0.8, 0.9$ when $1/(\lambda T) = 300$, $c_{i2}/c_{i1} = 10$, $c_d T/c_{i1} = 100$. For example, when $p = 0.9$, the optimal number is $m^* = 24$ and $C(m^*;T) = 1.95$. This also shows the same tendency as Figure 5.3.

5.6 Conclusions

We have proposed the optimal inspection policies for a system with two types of inspection. There might exist some failures in many actual systems which can not be detected by type-1 inspection and can be detected only by type-2 inspection. This assumption would be realistic, and the model is also simple. Further, it is easy to understand the results obtained and techniques used in this paper.

Using the inspection policy in reliability theory, we have derived the total expected cost until the detection of failure and the expected cost per unit of time. We have discussed analytically the optimal inspection policies which minimize the expected costs. We have given numerical examples when the failure time distribution is exponential. These formulations and results would be applied to other real systems such as digital circuits by suitable modifications.

Chapter 6

Optimal Replacement Policy for a System with Two Types of Inspection

This chapter considers a replacement policy for the same system with two types of inspection in Chapter 5: Type-1 inspection is done so frequently more than type-2 inspection, because the loss cost for one check of type-1 inspection is lower than that of type-2 inspection. On the other hand, there exist some failures which can not be detected and can be detected only by type-2 inspection. Further, the system is replaced at the specified N-th type-2 inspection. The expected cost per unit of time is analytically obtained, and an optimal number to perform type-1 inspection until the next type-2 inspection is derived. Numerical examples are given when the failure time distribution is exponential.

6.1 Introduction

This chapter treats an extended model in Chapter 5, where a system is checked by two-types of inspection: Type-1 inspection checks the system more frequently than type-2 inspection, since the cost for one check of type-1 inspection is lower than that of type-2 inspection. However, there exist some failures which can not be detected by type-1 inspection and can be detected only by type-2 inspection. Further, we suppose the system is replaced at the N-th type-2 inspection, and thereafter, it becomes like new. That is, the system operates until either the N-th type-2 inspection or the time at which its failure is detected by type-1 or type-2 inspection. Kodo *et al.* (1999) considered the optimal maintenance policy for a phased array radar, which is replaced at either specified number of inspection or at time when the total of failed elements have exceeded a specified number.

The inspection policy with replacement is applied to such a model: Type-1 inspection checks a system at periodic times jT $(j = 1, 2, \ldots, Nm)$, and type-2 inspection checks it at periodic time kmT $(k = 1, 2, \ldots, N)$, where m is the number to perform type-1 inspection until the next type-2 inspection, and T $(0 < T < \infty)$ is constant and the interval of type-1 inspection. When the system fails, its failure can be detected by type-1 inspection with probability p. On the other hand, some failures can not be detected by type-1 inspection with probability $1 - p$. The system operates until the time NmT or the time that its failure is detected by inspection, whichever occurs first.

We introduce the loss cost for the time elapsed between a failure and its detection. Then, the mean time and total expected cost to replacement, and the expected cost per unit of time are derived. An optimal number m^* which minimizes the expected cost is analytically derived for given T and N. Finally, numerical examples are given when the failure time distribution is exponential.

6.2 Model and Assumptions

We consider a system which is checked periodically by type-1 inspection or type-2 inspection, and is replaced at the N-th type-2 inspection. It is assumed that type-

2 inspection can detect any failure which can not be detected by type-1 inspection. Further, when the system is checked by the finite number of type-2 inspection, it is replaced and becomes like new.

For this model, we make the following assumptions:

(i) The system is checked by two types of inspection; type-1 or type-2 inspection. The system is replaced at the specified N-th type-2 inspection or the time when its failure is detected by inspection, whichever occur first. Any failure does not occur between the first failure and the next inspection. If the failure is detected then the system is maintained and is as good as new.

(ii) The system is checked periodically by two types of inspection. Type-1 inspection is performed at periodic times jT ($j = 1, 2, \dots, Nm$) for some specified T ($0 < T < \infty$). Type-2 inspection is performed at periodic times kmT ($k = 1, 2, \dots, N$), where m ($m = 1, 2, \dots$) is the number to perform type-1 inspection until the next type-2 inspection, *i.e.*, type-2 inspection is done at every m times of type-1 inspection.

(iii) The failure time has a general distribution $F(t)$ with finite mean $1/\lambda$, where $\overline{F}(t) \equiv 1 - F(t)$.

(iv) When the system fails, its failure is detected in the following way: The failure can be detected by type-1 inspection with probability p ($0 < p \leq 1$) and type-2 inspection. On the other hand, the failure can not be detected by type-1 inspection with probability $1 - p$, and can be detected only by type-2 inspection, *i.e.*, type-2 inspection can detect any failure.

(v) A cost c_{i1} is the cost for one check of type-1 inspection, c_{i2} is the cost for one check of type-2 inspection, that is, the inspection cost at time kmT includes two

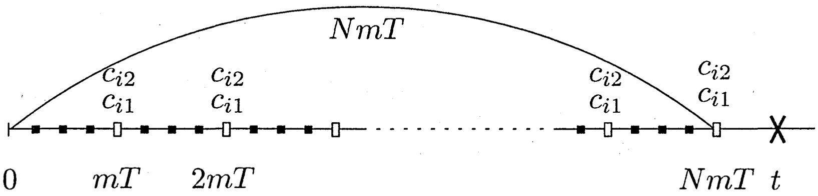

Figure 6.1: Diagram in case of $NmT < t$.

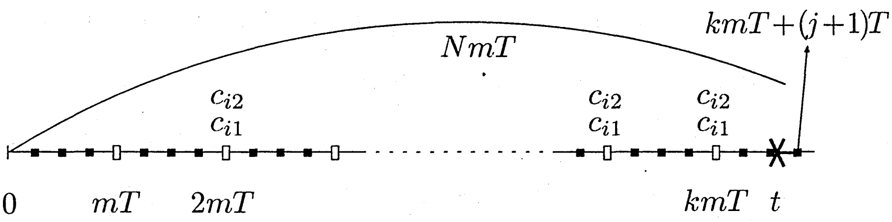

Figure 6.2: Diagram in case of $kmT + j < t < kmT + (j + 1)T$.

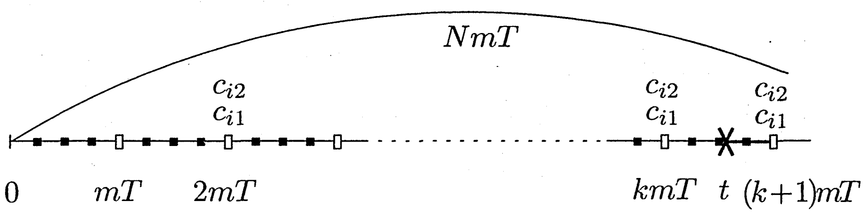

Figure 6.3: Diagram in case of $kmT < t \leq (k + 1)T \leq NmT$.

costs of type-1 inspection and type-2 inspection. Further, c_d is the loss cost per unit of time for the time elapsed between a failure and its detection.

When the system fails at time t, this model is classified into the following three cases:

1) Case of $NmT < t$

The mean time to replacement is

$$\int_{NmT}^{\infty} NmT\, dF(t) = NmT\overline{F}(NmT). \tag{6.1}$$

Further, the expected cost to replacement is

$$\int_{NmT}^{\infty} (Nmc_{i1} + Nc_{i2} + c_r)\, dF(t) = (Nmc_{i1} + Nc_{i2} + c_r)\overline{F}(NmT). \qquad (6.2)$$

2) Case that the failure is detected at time t $(kmT+jT < t \le kmT+(j+1)T \le NmT)$ with probability p

The mean time to replacement is

$$\sum_{k=0}^{N-1}\sum_{j=0}^{m-1}\int_{kmT+jT}^{kmT+(j+1)T} [kmT + (j+1)T]\, dF(t) = T\sum_{k=0}^{Nm-1}\overline{F}(kT) - NmT\overline{F}(NmT). \quad (6.3)$$

Further, the total expected cost to replacement is

$$\sum_{k=0}^{N-1}\sum_{j=0}^{m-1}\int_{kmT+jT}^{kmT+(j+1)T} \{[km + (j+1)]c_{i1} + kc_{i2} + [kmT + (j+1)T - t]c_d + c_r\}\, dF(t)$$

$$= c_{i1}\sum_{k=0}^{Nm-1}\overline{F}(kT) + c_{i2}\sum_{k=1}^{N}\overline{F}(kmT) - N(mc_{i1} + c_{i2})\overline{F}(NmT)$$

$$+ c_d\left[T\sum_{k=0}^{Nm-1}\overline{F}(kT) - \int_0^{NmT}\overline{F}(t)\, dt\right] + c_r F(NmT). \qquad (6.4)$$

3) Case that the failure is detected at time t $(kmT < t \le (k+1)T \le NmT)$ with probability $1-p$

The mean time to replacement is

$$\sum_{k=0}^{N-1}\int_{kmT}^{(k+1)mT} (k+1)mT dF(t) = mT\sum_{k=0}^{N-1}\overline{F}(kmT) - NmT\overline{F}(NmT). \qquad (6.5)$$

Further, the expected cost to replacement is

$$\sum_{k=0}^{N-1}\int_{kmT}^{(k+1)mT} \{(k+1)mc_{i1} + (k+1)c_{i2} + [(k+1)mT - t]c_d + c_r\}dF(t)$$

$$= (mc_{i1} + c_{i2})\left[\sum_{k=0}^{N-1}\overline{F}(kmT) - N\overline{F}(NmT)\right] + c_d\left[mT\sum_{k=0}^{N-1}\overline{F}(kmT) - \int_0^{NmT}\overline{F}(t)\, dt\right]$$

$$+ c_r F(NmT). \qquad (6.6)$$

Thus, the mean time $A(m;T,N)$ to replacement is obtained as the summation of equations (6.1), (6.3), (6.5) as follows:

$$A(m;T,N)=p\left[T\sum_{k=0}^{Nm-1}\overline{F}(kT)\right]+(1-p)\left[mT\sum_{k=0}^{N-1}\overline{F}(kmT)\right].$$

Similarly, the total expected cost $B(m;T,N)$ to replacement is obtained as the summation of equations (6.2), (6.4), (6.6) as follows:

$$B(m;T,N)=c_r+(c_{i1}+c_dT)\left[p\sum_{k=0}^{Nm-1}\overline{F}(kT)+(1-p)m\sum_{k=0}^{N-1}\overline{F}(kmT)\right]$$
$$+c_{i2}\left[p\sum_{k=1}^{N}\overline{F}(kmT)+(1-p)\sum_{k=0}^{N-1}\overline{F}(kmT)\right]-c_d\int_0^{NmT}\overline{F}(t)\,dt.$$

6.3 Optimal Inspection Policy

We seek an optimal number m^* of type-2 inspection which minimizes the expected cost $C(m;T,N)$ when the failure distribution is $F(t)=1-e^{-\lambda t}$. Then, the expected cost $C(m;T,N)$ is given by

$$C(m;T,N)\equiv\frac{B(m;T,N)}{A(m;T,N)}$$
$$=\frac{\left[\left\{(c_{i1}+c_dT)\left[\frac{p}{1-e^{-\lambda T}}+\frac{(1-p)m}{1-e^{-\lambda mT}}\right]+c_{i2}\frac{1-p(1-e^{-\lambda mT})}{1-e^{-\lambda mT}}-\frac{c_d}{\lambda}\right\}\times(1-e^{-\lambda NmT})+c_r\right]}{\left[\frac{p}{1-e^{-\lambda T}}+\frac{(1-p)m}{1-e^{-\lambda mT}}\right](1-e^{-\lambda NmT})T}$$
$$=c_d+\frac{c_i}{T}+\frac{c_{i2}\frac{1-p(1-e^{-\lambda mT})}{1-e^{-\lambda mT}}-\frac{c_d}{\lambda}+\frac{c_r}{(1-e^{-\lambda NmT})}}{\left[\frac{p}{1-e^{-\lambda T}}+\frac{(1-p)m}{1-e^{-\lambda mT}}\right]T}. \tag{6.7}$$

In particular, when $c_r=0$, letting $C(m+1;T,N)\geq C(m;T,N)$, we have

$$\frac{(m+1)(1-e^{-\lambda mT})-m(1-e^{-\lambda(m+1)T})}{1-p(1-e^{-\lambda mT})}\geq\frac{c_{i2}}{(c_{i2}p+c_d/\lambda)(1-p)}. \tag{6.8}$$

Letting denote the left-side hand of (6.8) by $L(m)$,

$$L(0) = 0,$$

$$L(\infty) = \frac{1}{1-p}.$$

Further,

$$(m+1)(1-e^{-\lambda mT}) - m(1-e^{-\lambda(m+1)T}) - m(1-e^{-\lambda(m-1)T}) + (m-1)(1-e^{-\lambda mT})$$

$$= m[e^{-\lambda(m-1)T} + e^{-\lambda(m+1)T} - 2e^{-\lambda mT}]$$

$$= me^{-\lambda(m-1)T}(1-e^{-\lambda T})^2 \geq 0.$$

Thus, the numerator of $L(m)$ is strictly increasing, and the denominator of $L(m)$ is decreasing with m. Therefore, $L(m)$ is strictly increasing from 0 to $1/(1-p)$, and hence, if $c_d/\lambda > c_{i2}$ then there exists an optimal m^* $(1 \leq m^* < \infty)$ which satisfies (6.8), independent of N.

Next, letting $C(m+1;T,N) - C(m;T,N) \geq 0$ in (6.7), we have

$$\frac{\begin{bmatrix} (c_{i2}p + c_d/\lambda)(1-p)[(m+1)(1-e^{-\lambda mT}) - m(1-e^{-\lambda(m+1)T})] \\ -c_{i2}[1-p(1-e^{-\lambda mT})] + Nc_{i2}V(m)/(1-p) \end{bmatrix}}{V(m)} \geq c_r + \frac{Nc_{i2}}{1-p}, \quad (6.9)$$

where

$$V(m) \equiv$$

$$\frac{\begin{bmatrix} (1-p)[(m+1)(1-e^{-\lambda mT})(1-e^{-\lambda N(m+1)T}) - m(1-e^{-\lambda(m+1)T})(1-e^{-\lambda NmT})] \\ -p(1-e^{-\lambda(m+1)T})(1-e^{-\lambda mT})(e^{-\lambda N(m+1)T} - e^{-\lambda NmT})/(1-e^{-\lambda T}) \end{bmatrix}}{(1-e^{-\lambda NmT})(1-e^{-\lambda N(m+1)T})}.$$

Letting denote the left-side hand of (6.9) by $L_r(m)$,

$$L_r(0) \equiv \lim_{m\to 0} L_r(m) = 0,$$

$$L_r(\infty) \equiv \lim_{m\to\infty} L_r(m) = \frac{c_d}{\lambda} + \left[\frac{N}{1-p} - (1-p)\right] c_{i2}.$$

Table 6.1: Optimal number m^* to minimize $C(m; T, N)$ for c_{i2}, $1/(\lambda T)$ and $c_d T$ when $p = 0.9$ and $c_r = 0$.

c_{i2}	$1/(\lambda T) = 300$			$1/(\lambda T) = 600$		
	$c_d T$					
	100	500	1000	100	500	1000
1	8	4	3	11	5	4
2	11	5	4	16	7	5
5	18	8	6	26	11	8
10	25	11	8	36	16	11
15	31	14	10	44	20	14
20	36	16	11	51	23	16
25	40	18	13	57	26	18
30	44	20	14	62	28	20

Therefore, if $c_d/\lambda + [N/(1-p) - (1-p)]c_{i2} \geq c_r$, there exists optimal m^* which minimizes $C(m; T, N)$.

6.4 Numerical Examples

We compute numerically an optimal number to perform type-1 inspection until the next type-2 inspection, *i.e.*, we compute numerically an optimal number m^* which minimizes the expected cost $C(m; T, N)$ when $F(t) = 1 - e^{-\lambda t}$.

Table 6.1 presents the optimal number m^* which minimizes the expected cost $C(m; T, N)$ for $1/(\lambda T) = 300$, 600, $c_d T = 100$, 500, 1000 and $c_{i2} = 1$, 2, 5, 10, 15, 20, 25, 30 when $p = 0.9$ and $c_r = 0$. It is shown that optimal number m^* increases as c_{i2} or $1/(\lambda T)$ increases and $c_d T$ decreases. This indicates that if the cost c_{i2} of type-2 inspection is small, it would be better to perform type-2 inspection early. For example, when $c_{i2} = 15$, $1/(\lambda T) = 300$, $c_d T = 100$ and $p = 0.9$, the optimal number is $m^* = 31$. That is, when the interval of type-1 inspection is $T = 1$ day, type-2 inspection should be performed almost every one month.

Table 6.2: Optimal number m^* to minimize $C(m; T, N)$ for p, $1/(\lambda T)$ and $c_d T$ when $c_{i2} = 10$ and $c_r = 0$.

p	$1/(\lambda T) = 300$			$1/(\lambda T) = 600$		
	$c_d T$					
	100	500	1000	100	500	1000
0.5	13	6	4	19	9	6
0.6	14	7	5	20	9	7
0.7	16	7	5	23	10	7
0.8	19	8	6	27	12	8
0.9	25	11	8	36	16	11
1.0	∞	∞	∞	∞	∞	∞

Table 6.3: Optimal number m^* to minimize $C(m; T, N)$ for N, $1/(\lambda T)$, and $c_d T$ when $c_{i2} = 10$, $c_r = 100$ and $p = 0.9$.

N	$1/(\lambda T) = 300$			$1/(\lambda T) = 600$		
	$c_d T$					
	100	500	1000	100	500	1000
1	79	36	25	112	51	36
2	59	27	19	83	38	27
3	50	23	16	71	32	23
4	45	20	14	64	29	20
5	41	19	13	59	27	19
10	33	15	11	48	22	15
15	30	14	10	44	20	14
20	28	13	9	41	19	13
25	27	13	9	40	18	13
35	26	12	9	38	17	12
40	26	12	9	37	17	12
45	25	12	8	36	17	12
50	25	12	8	36	17	12

Table 6.2 shows an optimal number m^* which minimizes the expected cost $C(m;T,N)$ for $1/(\lambda T) = 300$, 600, $c_dT = 100$, 500, 1000 and $p = 0.5$, 0.6, 0.7, 0.8, 0.9, 1.0 when $c_{i2} = 10$. It is shown that the optimal number m^* increases with probability p. This indicates when p is small, it would be better to perform type-2 inspection early.

Table 6.3 shows the optimal number m^* which minimizes the expected cost $C(m;T,N)$ for $1/(\lambda T) = 300$, 600, $c_dT = 100$, 500, 1000 and $N = 1$, 2, 3, 4, 5, 10, 15, 20, 25, 30, 35, 40, 45, 50 when $c_{i2} = 10$, $c_r = 100$ and $p = 0.9$. This indicates that the optimal m^* decreases with N, *i.e.*, if N is large, it would be better to perform type-2 inspection shorter. Further, when N increase, the increment of m^* decrease.

6.5 Conclusions

We have proposed the optimal replacement policies for a system with two types of inspection. The system is replaced at the finite N-th type-2 inspection, and after that, it becomes like new. There might exist some failures which can not be detected by type-1 inspection and can be detected only by type-2 inspection. This assumption would be realistic, and the model is also simple.

Using the optimal policy in reliability theory, we have derived the expected cost per unit of time and have discussed analytically the optimal inspection policy which minimize it. Numerical examples have been given when the failure time distribution is exponential.

Chapter 7

Conclusions

This thesis has studied the optimal inspection and maintenance policies for high reliable systems. We have suggested several useful models where systems such as digital control devices are checked by inspection.

To prevent the affect of failures, inspection and maintenance should be done so frequently. However, it might incur much loss cost and laborious work to perform inspection and maintenance. We have obtained the optimal policies analytically by making a trade-off between the loss cost of failures and the cost of inspection. We have considered one cycle as the time from the beginning of system operation to the detection of failure. Using the reliability theory, we have obtained the mean time and the total expected cost of one cycle, and the expected cost per unit of time. Further, we have derived analytically the optimal inspection and maintenance schedules which minimize these expected costs, and have given numerical examples of each model and have evaluated them to understand the results easily.

As an application for above results, we have mainly discussed how to determine the schedules of self-diagnosis for digital control devices. However, the results would be applied to other inspection and maintenance policies for actual systems such as industrial or power plants, aircrafts, and so on.

Some valuable contributions to the study of inspection and maintenance policies in reliability theory have been made as follows:

In Chapter 2, we have derived the optimal inspection policies for a two-unit system: The system firstly operates as a two-unit system and is checked by comparison-checking. When one unit fails, the system operates as a single-unit system and is checked periodically by self-diagnosis. We have introduced the costs of one check for a two-unit system and for a single-unit system. In this model, we have proposed two models of comparison-checking model: (1) Continuous comparison-checking model: When the system operates as two-unit system, it is checked continuously by comparison-checking. Thus, failures of a unit are detected immediately. (2) Periodic comparison-checking model: When the system operates as a two-unit system, it is checked periodically by comparison-checking. It is assumed for simplicity that the intervals and periodic self-diagnosis for a single-unit system are the same. We have derived the expected costs for each model, and have discussed the optimal inspection policies which minimize them. When the failure time has an exponential distribution, numerical examples has been shown for several parameters.

In Chapter 3, we have given the optimal inspection policies for a system with self-testing: The system can detect its failure during its operating state without external inspection. However, the system has the latency of detection by self-testing, *i.e.*, some failures might not be detected rapidly. Therefore, for the system required high reliability, it should be checked by external inspection at scheduled times. Thus, if the system fails, then its failure is detected by self-testing or at the next periodic inspection, whichever occur first. It has been shown that the self-detection rate plays an important role for deriving optimal policies. We have proposed the periodic inspection model and sequential inspection model. We have shown the optimal policies which minimize the expected costs, and have computed the numerical examples for each model.

In Chapter 4, we have given the optimal maintenance and inspection policies for a finite interval: In actual fields, most systems have a finite span of use. In this chapter, we have used the partition method for this problem, where a finite interval is divided into equal parts of maintenance or inspection. Optimal policies which minimize the expected costs of periodic replacement with minimal repair, block replacement, simple replacement and inspection policy for a finite interval have been derived. It has been shown that three replacement models are summarized on a general form. Further, we have given numerical examples of each model and have evaluated them to understand the result easily.

In Chapter 5, we have given the optimal inspection policies for a system which is checked by two types of inspection: Type-1 inspection has lower cost for one check than the cost of type-2 inspection. Hence, type-1 inspection checks the system more frequently than type-2 inspection. However, there exist some failures which can not be detected by type-1 inspection and can be detected only by type-2 inspection. We have derived analytically the optimal number to check type-1 inspection until the next type-2 inspection. It has shown from numerical examples that the optimal number of type-1 inspection until the next type-2 inspection decreases with probability $1 - p$.

In Chapter 6, we have given the replacement policy for the same inspection model as Chapter 5: Type-1 inspection has the lower cost for one check than the cost of type-2 inspection. Hence, type-1 inspection checks the system more frequently than type-2 inspection. However, there exist some failures which can not be detected by type-1 inspection and can be detected only by type-2 inspection. In this chapter, we have supposed the system is replaced at the specified N-th type-2 inspection, or the time at which its failure is detected by inspection, whichever occur first. We have derived analytically the optimal number to check type-1 inspection until the next type-2 inspection.

In this thesis, we have studied the optimal inspection and maintenance policies for high reliable systems such as digital control devices. We have analyzed their reliability characteristics, and have established new and adapted policies. Using the results and techniques derived in this thesis, these policies would be modified and developed, and be applied actually to many practical systems needed for inspection.

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Chapter 2

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Chapter 4

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Chapter 5

1. Satoshi Mizutani, Toshio Nakagawa and Kodo Ito:
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   Proceeding of *Ninth ISSAT International Conference on Reliability and Quality in Design*, pp. 46-49, August 2003.

Chapter 6

1. Satoshi Mizutani and Toshio Nakagawa:
   "Optimal Replacement Policy for a System with Two Types of Inspection"
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