On the transitivity of the regularity of potential kernels by the relative domination principle

Dedicated to Professor Masanori KISHI on the occasion of his 60th birthday

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ポテンシァル核の正則性の相対優越原理による推移性について

樋口　功

Let G and N be two continuous function-kernels on a locally compact Hausdorff space X. Suppose that G satisfies the relative domination principle with respect to N. We pose the following three problems:

(1) For a lower semi-continuous function, does the N-superharmonicity imply also the G-superharmonicity?

(2) Does the regularity of N transit to the regularity of G?

(3) Does the N-thinness at infinity of a closed set F in X imply also the G-thinness of F at infinity?

In the present paper, we shall give the affirmative answers to the above three problems.

§1. Preliminaries

Let X be a locally compact Hausdorff space with countable basis. A non-negative function $G=G(x,y)$ on $X\times X$ is called a continuous function-kernel on X if $G(x,y)$ is continuous in the extended sense on $X\times X$, finite except for the diagonal set of $X\times X$ and $0<G(x,x)\leqq+\infty$ for any $x\in X$. The kernel $\check{G}$ defined by $\check{G}(x,y)=G(y,x)$ is called the adjoint kernel of G.

We denote by M the set of all positive measures on X. The potential $G\mu(x)$ and the adjoint potential $\check{G}\mu(x)$ of $\mu\in M$ are defined by

$$G\mu(x) = \int G(x,y)d\mu(y) \quad \text{and} \quad \check{G}\mu(x) = \int \check{G}(x,y)d\mu(y)$$

respectively. The G-energy of $\mu\in M$ is defines by $\int G\mu(x)d\mu(x)$.

Put

$$M_0 = \{\mu\in M \ ; \ \text{support } S\mu \text{ of } \mu \text{ is compact}\},$$

$$E = E(G) = \{\mu\in M_o \ ; \ \int G\mu(x)d\mu(x) < +\infty\},$$

$$F_c(G) = \{\mu\in E(G) \ ; \ G\mu(x) \text{ is finite and continuous on } X\}.$$

A Borel measurable set B is said to be G-negligible if $\mu(B)=0$ for every $\mu\in E(G)$. We say that a property holds G-nearly everywhere on a subset A of X (written symply G-n.e. on A), when it holds on A except for a G-negligible set.

A non-negative lower semi-continuous function $u(x)<+\infty$ G-n.e. on X is said to be G-superharmonic when for any $\mu\in E(G)$, the inequality $G\mu(x)\leqq u(x)$ G-n.e. on $S\mu$ implies the same inequality on the whole space X.

We denote by $S(G)$ the totality of G-superharmonic functions on X and by $P_{M_o}(G)$ (resp. $P_E(G)$) the totality of G-potentials of measures in $M_o(G)$ (resp. $E(G)$).

The potential throretical principles are stated as follows.

(I) We say that G satisfies the *domination principle* and write symply $G\prec G$ when $P_{M_o}(G)\subset S(G)$.

(II) We say that G satisfies the *maximum principle* when $1\in S(G)$.

(III) We say that G satisfies the *complete maximum principle* when, for any non-negative number a, $P_{M_o}(G)\cup\{a\}\subset S(G)$.

(IV) We say that G satisfies the *relative domination principle with respect to* N and write symply $G\prec N$ when $P_{M_o}(N)\subset S(G)$.

(V) We say that G satisfies the *transitive domination principle with respect to* N and write symply $G\sqsubset N$ when, for $\mu\in E(G)$ and $\nu\in M_o$, an inequality $G\mu(x)\leqq G\nu(x)$ on $S\mu$ implies the inequality $N\mu(x)\leqq N\nu(x)$ on X.

(VI) We say that G satisfies the *balayage principle* if, for any compact set K and any $\mu\in M_o$, there exists a measure μ' in M_o, called a *balayaged measure* of μ on K, supported by K satisfying

$$G\mu'(x) = G\mu(x) \qquad G\text{-n.e. on } K,$$
$$G\mu'(x) \leqq G\mu(x) \qquad \text{on } X.$$

(VII) We say that G satisfies the *equilibrium principle* if, for any compact set K, there exists a measure γ in M_o, called a *equilibrium measure* of K, supported by K satisfying

$$G\gamma(x) = 1 \qquad G\text{-n.e. on } K,$$
$$G\gamma(x) \leqq 1 \qquad \text{on } X.$$

(VIII) We say that G satisfies the *relative balayage principle with respect to* N and write symply $G\preccurlyeq N$ if, for any compact set K and any $\mu\in M_o$, there exists a measure μ'' in M supported by K satisfying

$$G\mu''(x) = N\mu(x) \qquad G\text{-n.e. on } K,$$
$$G\mu''(x) \leqq N\mu(x) \qquad \text{on } X.$$

(IX) We say that G satisfies the *continuity principle* if, for $\mu \in M_o$, the finite continuity of the restriction of $G\mu(x)$ to $S\mu$ implies the finite continuity of $G\mu(x)$ on the whole space X.

REMARK 1. On the relations between the potential theoretical principles, the following results are well known.

(1) If a continuous function-kernel G satisfies the domination principle, then both G and $\check{G}$ satisfy the continuity principle (cf. [5]).

(2) For a continuous function-kernel G, the following four statements are equivalent (cf. [5] and [7]):

(a) G satisfies the domination principle.
(b) $\check{G}$ satisfies the domination principle.
(c) G satisfies the balayage principle.
(d) $\check{G}$ satisfies the balayage principle.

(3) Let G and N be two lower semi-continuous function-kernels on X. Suppose that both G and $\check{G}$ satisfy the continuity principle. Then the following three statements are equivalents (cf. [7]).

(a) $G \prec N$. (b) $\check{G} \sqsubset \check{N}$. (c) $G \preccurlyeq N$.

§2. Transitivity of superharmonicity

First we shall prove that the superharmonicity of functions is a transitive property by the relative domination priciple.

THEOREM 1. *Let G and N be two continuous function-kernels on X. Suppose that G satisfies the relative domination principle with respect to N. Then we have $S(N) \subset S(G)$, namely, a lower semi-continuous function is G-superharmonic if it is N-superharnomic.*

It is well known that when $G \prec N$, G satisfies the continuity principle but $\check{G}$ does not necessarily satisfy the continuity principle. Therefore we can not use the theorem of M.KISHI directly (cf. (3) in REMARK 1).

For the proof of THEOREM 1, it suffices to recollect the following

LEMMA 1 (cf. [2]). *Let G and N be two continuous function-kernels on X and K be a non-negative lower semi-continuous function-kernel on X. If $G \prec N \prec K$, then $G \prec K$, namely the relation $\prec$ is transitive.*

§3. Transitivity of regularity of function-kernels

For a Borel function u and a closed set F, we put

$$S_u^F(G) = \{ v \in S(G) \ ; \ v(x) \geqq u(x) \ G\text{-n.e. on } F \}.$$

The *G-reduced function of u on F* and the *G-reduced function of u at infinity δ on F* are defined respectively by

$$R_G^F(u)(x) = \begin{cases} \inf\{v(x)\ ;\ v \in S_u^F(G)\} & \text{if } S_u^F(G) \neq \phi \\ +\infty & \text{if } S_u^F(G) = \phi, \end{cases}$$

$$R_G^{F,\delta}(u)(x) = \inf_{\omega \in \mathscr{G}_0} R_G^{F \cap C\omega}(u)(x),$$

where we denote by $\mathscr{G}_0$ the totality of all relatively compact open sets.

For any closed set F, the subset $S_o(F;G)$ of $S(G)$ is defined by

$$S_o(F;G) = \{u \in S(G)\ ;\ R_G^{F,\delta}(u)(x) = 0 \ G\text{-n.e. on } X\}.$$

A continuous function-kernel G is said to be *regular* when

$$P_{M_0}(G) \subset S_o(X;G),$$

and is said to be *regular on a closed set F* when

$$P_{M_0}(G) \subset S_o(F;G).$$

REMARK 2. It is well known that the following five statements are equivalent each other (cf. [3] and [6]).

(1) G is regular.

(2) $P_E(G) \subset S_o(X;G)$.

(3) $P_{F_c}(G) \subset S_o(X;G)$.

(4) For any measure $\mu \in F_c(\check{G})$, $R_G^{X,\delta}(\check{G}\mu)(x)=0$ everywhere on X.

(5) $\check{G}$ is regular.

The regularity of function-kernels is a transitive property by the relative domination principle, namely we have the following

THEOREM 2. *Let G and N be two continuous function-kernels on X such that both G and N satisfy the domination principle and that every open set $\omega(\neq \phi)$ is non-negligible with respect to G and N. Suppose that G satisfies the relative domination principle with respect to N. Then G is regular if N is regular.*

For the proof of THEOREM 2, first we remember the following lemma obtained in [4].

LEMMA 2. *Let G be a continuous function-kernel on X satisfying the domination principle. Then for any $u \in S(G)$ and any compact set K, there exists a sequence $(\mu_n)_{n=1}^{+\infty} \subset E(G)$ such that*

$$S\mu_n \subset K,$$

$$G\mu_n(x) \leqq G\mu_{n+1}(x) \leqq u(x) \text{ on } X,$$

$$\lim_{n\to+\infty} G\mu_n(x) = u(x) \text{ } G\text{-n.e. on } K.$$

Further $(\mu_n)_{n=1}^{+\infty}$ *is vaguely bounded and its vague accumulation point* μ' *fulfills*

$$G\mu'(x) = R_G^K(u)(x) \text{ on } X.$$

LEMMA 3. *Let G and N be two continuous function-kernels on X satisfying the domination principle. Suppose that G satisfies the relative domination principle with respect to N. Then we have*

$$S_o(X,N) \subset S_o(X,G).$$

PROOF. Let $u \in S_o(X,N) \subset S(G)$. For any $\omega \in \mathscr{G}_o$ and any compact set K, there exists, by virtue of LEMMA 2, a sequence $(\mu_n)_{n=1}^{+\infty} \subset E(G)$ satisfying

$$S\mu_n \subset C\omega \cap K,$$

$$G\mu_n(x) \leqq G\mu_{n+1}(x) \leqq u(x) \text{ on } X,$$

$$\lim_{n\to+\infty} G\mu_n(x) = u(x) \text{ } G\text{-n.e. on } C\omega \cap K.$$

And a vague accumulation point μ' of $(\mu_n)_{n=1}^{+\infty}$ fulfills

$$G\mu'(x) = R_G^{C\omega\cap K}(u)(x) \text{ on } X.$$

For any $\nu \in F_c(\check{G})$, we denote by $\check{\nu}'_{C\omega\cap K}$ a balayaged measure of ν on $C\omega\cap K$ with respect $\check{G}$. Then we have

$$\begin{aligned}\int R_G^{C\omega\cap K}(u)d\nu &= \int G\mu' d\nu \leqq \int \lim_{n\to+\infty} G\mu_n d\nu = \lim_{n\to+\infty} \int G\mu_n d\nu \\ &= \lim_{n\to+\infty} \int \check{G}\nu d\mu_n = \lim_{n\to+\infty} \int \check{G}\check{\nu}'_{C\omega\cap K}\, d\mu_n \\ &= \lim_{n\to+\infty} \int G\mu_n d\check{\nu}'_{C\omega\cap K} \leqq \int u d\check{\nu}'_{C\omega\cap K} \leqq \int R_N^{C\omega\cap K}(u) d\check{\nu}'_{C\omega\cap K} = \int R_N^{C\omega\cap K}(u)d\nu,\end{aligned}$$

where the last two inequalities are obtained by the fact that $\check{\nu}'_{C\omega\cap K}$ is in $E(N)$ and that $R_N^{C\omega\cap K}(u)$ is in $S(N)$ and hence in $S(G)$ (see THEOREM 1). Letting K and ω tend to X, we have

$$\int R_G^{X,\delta}(u)d\nu \leqq \int R_N^{X,\delta}(u)d\nu = 0.$$

Consequently, u belongs to $S_o(X,G)$ and hence we have

$$S_o(X,N) \subset S_o(X,G).$$

PROOF OF THEOREM 2. Suppose that N is regular.

By our assumption that any open set $\omega \neq \phi$ is non-negligible, there is, for any $\mu \in F_c(G)$, a measure $\nu \in E(N)$ such that $G\mu(x) \leqq N\nu(x)$ on $S\mu$. Then, by virtue of the relative domination principle, we have $G\mu(x) \leqq N\nu(x)$ on X.

The kernel N being regular, $N\nu(x)$ belongs to $S_o(X,N)$. From the relation $S_o(X,N) \subset S_o(X,G)$ in LEMMA 3, it follows that both $N\nu(x)$ and $G\mu(x)$ belong to $S_o(X,G)$.

Therefore, by (3) in REMARK 2, we can asserts that G is regular.

This completes the proof.

§3. Transitivity of thinness at infinity of closed set

A closed set F in X is said to be *G-thin at infinity* δ when 1 is in $S_o(F,G)$.

Finally we prove that the thinness at infinity of a closed set is also a transitive property by the relative domination principle.

THEOREM 3. *Let G and N be two continuous function-kernels on X satisfying the complete maximum principle. Suppose that G satisfies the relative domination principle with respect to N. Then a closed set F is G-thin at infinity δ if it is N-thin at δ.*

PROOF. We can prove, by an argument similar to LEMMA 2, that $S_o(F,N)$ is contained in $S_o(F,G)$ when G and N satisfy the domination principle and $G \prec N$.

The complete maximum principle asserts that $1 \in S(G) \cap S(N)$. F being N-thin at δ, 1 is in $S_o(F,N)$ and hence in $S_o(F,G)$.

Therefore F is G-thin at δ.

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