Exact Solution of the Two-Dimensional Toda Lattice

Fukuko YUASA* and Éi Iti TAKIZAWA

二次元戸田格子の厳密解

湯 浅 富久子・瀧 澤 英 一

The authors present an exact solution of the two-dimensional Toda lattice equation. The solution obtained here is a cnoidal wave, which is periodic function with regard to the angular variable and lattice site number n.

§ 1. Preliminaries

The Toda lattice[1,2] found in 1967 seems to us as one of gems brilliantly radiating in the field of soliton dynamics. Many investigations have been also accumulated to solve nonlinear problems, such as Hirota's Direct method[3,4], the inverse scattering scheme[5-7], the Lax formalism[8,9], the Bäcklund transformation[10], etc.

A generalization of the Toda lattice to the multi-(space)-dimensional systems has been also tried. For examples, the two-dimensional Toda lattice (2DTL) was proposed by Mikhailov[11] in 1979 and solved in the axially symmetric case by Nakamura[12] in 1983. This system is reconsidered in a series of our papers by the generalized recurrence formulae[13-15], or essentially by the Bäcklund transformation[16,17,20], and its solutions were given by means of cylindrical functions[12,15,16]. The N-soliton solution of the 2DTL equation in rectangular coordinates was given by Hirota[18,19] in a form of the Casorati determinant, and it was also shown[20] that the N-soliton solution of the 2DTL equation can be derived from the (N-1)-soliton solution by means of the generalized recurrence formulae.

As for the three-dimensional generalization of the Toda lattice system, travelling wave solutions[21] have been already given by the method of dimensional reduction[22]. Recently, Nakamura[23] found a new type of exact solutions of the three-dimensional Toda lattice (3DTL) in terms of associated Legendre functions. He also showed that the Bessel-type solution of the 2DTL equation can be obtained as a limitting case of the Legendre-type solution of the 3DTL equation. Hirota and Nakamura[24] treated the Toda molecule (the Toda lattice with finite length) in the two-dimensional case, and Hirota[25] considered a discrete version of the Toda molecule, obtaining exact solutions in a form of determinant.

This paper aims to obtain an exact solution of the 2DTL equation as a periodic function of the peripheral angle.

§ 2. The Two-Dimensional Toda Lattice

The one-(time)-dimensional Toda lattice equation reads:

$$\frac{d^2}{dt^2}\log V_n(t) = V_{n+1}(t) - 2V_n(t) + V_{n-1}(t), \qquad (1)$$

* National Laboratory for High Energy Physics, Tsukuba-shi, Ibaraki-ken, Japan

with replacement :

$$V_n(t) = \exp[-\{u_{n+1}(t)-u_n(t)\}] \ , \qquad (2)$$

where $u_n(t)$ is displacement of particle at lattice site No. n, and t stands for time variable. Equation (1) is generalized to the Toda lattice equation in the two-dimensional Euclidean space (x_1, x_2) :

$$\Delta \log V_n(x_1, x_2) = V_{n+1}(x_1, x_2)-2V_n(x_1, x_2)+V_{n-1}(x_1, x_2), \qquad (3)$$

with Laplacean operator :

$$\Delta = \sum_{k=1}^{2} \frac{\partial^2}{\partial x_k{}^2} \cdot \qquad (4)$$

In the two-dimensional polar coordinates (ρ, ϕ) :

$$x_1 = \rho\cdot\cos\phi, \quad \text{and} \quad x_2 = \rho\cdot\sin\phi, \qquad (5)$$

for $\underline{0 \leqq \rho < +\infty}$ and $\underline{0 \leqq \phi < 2\pi}$, equation (3) can be written as :

$$\Delta \log V_n(\rho, \phi) = V_{n+1}(\rho, \phi)-2V_n(\rho, \phi)+V_{n-1}(\rho, \phi) \qquad (6)$$

with the two-dimensional Laplacean operator :

$$\Delta = \sum_{k=1}^{2} \frac{\partial^2}{\partial x_k{}^2} = \frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\cdot$$

If we want to take a solution of the form :

$$V_n(\rho, \phi) = \rho^{-2}\cdot U_n(\phi), \qquad (7)$$

then the angular part $U_n(\phi)$ of solution of eguation (6) satisfies :

$$\frac{d^2}{d\phi^2} \log U_n(\phi) = U_{n+1}(\phi)-2U_n(\phi)+U_{n-1}(\phi), \qquad (8)$$

which is nothing but the one-dimensional Toda lattice (1DTL) equation for the peripheral angle variable ϕ.

§ 3. Cnoidal Wave Solution

A solution of equation (8) under periodic boundary condition:

$$U_n(\phi) = U_n(\phi+2\pi), \qquad (9)$$

can be written[2] as

$$U_n(\phi) = 1+\frac{K}{\pi}\cdot\frac{\partial}{\partial\phi} Z\left[2K\left(\frac{\phi}{2\pi}+\frac{n}{A}\right)\right] = 1+\left[\frac{K}{\pi}\right]^2 \left[dn^2\left\{2K\left(\frac{\phi}{2\pi}+\frac{n}{A}\right)\right\}-\frac{E}{K}\right], \qquad (10)$$

with

$$\frac{K}{\pi} = \left[\frac{1}{sn^2(2K/A)}-1+\frac{E}{K}\right]^{-1/2},$$

A an arbitrary constant, and K and E complete elliptic integrals of the first and the second kinds, respectively :

$$K = K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}, \quad \text{and } E = E(k) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\theta}\, d\theta.$$

Here k is the modulus $(0<k<1)$, $Z(u)=Z(u,k)$ Zeta function, and $dn(u) = dn(u,k)$ Jacobi's dn-function.

Thus the final form of solution of equation (6) reads :

$$V_n(\rho, \phi) = \rho^{-2}\cdot U_n(\phi), \qquad (11)$$

with expression (10).

For the limitting case : $n \to +\infty$, equation (8) reads :

$\partial^2 \log U_\infty(\phi)/\partial\phi^2 = 0$, and we have a trivial solution :

$$U_\infty(\phi) = \exp[\alpha\phi+\beta], \qquad (12)$$

with constants α and β. The solution V_∞ with (12) is seen to be

$$V_\infty(\rho, \phi) = \rho^{-2}\cdot\exp[\alpha\phi+\beta].$$

The periodic boundary condition (9) demands $\alpha = i\gamma/2\pi$, with any integer γ and the imaginary unit $i = = \sqrt{-1}$.

§ 4. Remarks

It is known at present that the 2DTL equation has two kinds of solutions. Namely, one is expressed by Bessel-type function[12)–17)] and the other is given in a form of the Casorati determinant[18,19)]. The former is an axially symmetric solution depending on ρ and n, while the latter depends on x_1, x_2, and n, being not always axially symmetric. Both of them are obtained essentially by solving the 2DTL bilinear equation.

Here, in this paper, the authors presented a quite different kind of solution from the two kinds cited above. The present solution is expressed by an elliptic function dn, which is periodic with regard to ϕ (peripheral angle variable) and n. This shows a sharp contrast to the solution of the 1DTL equation, which depends on x_1 (linear coordinate) and n.

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