愛知工業大学研究報告
第21号Ａ　昭和61年

Remarks on High Order Differential Theoretic Characterization of Regular Local Rings

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正則局所環の高次微分論的特徴付けについての注意

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In this paper, we assume that R is a reduced noetherian local ring with coefficient field K of characteristic 0. For any positive integer n, we shall define the module of n-order differentials of R and will be denoted by $D^n_K(R)$. The main result of this paper is to prove that R is a regular local ring if and only if, for any positive integer n, $D^n_K(R)$ is a formally projective R-module. The case $n=1$ was proved in [1].

1. Preliminaries. In this paper all rings are assumed to be commutative rings with unit elements, all topological rings and modules are assumed to have linear topologies, that is, there exist fundamental systems of neighborhoods of (0) formed of ideals or submodules, respectively.

Let P be a ring. Let A be a P-algebra with the structure homomorphism $\varphi: P \to A$. We denote by δ the A-algebra homomorphism $A \otimes_P A \to A$ which is defined by $\delta(\sum_i a_i \otimes b_i) = \sum_i a_i b_i$ and denote by $I_P(A)$ the kernel of δ. Then it is easy to see that $I_P(A)$ is the ideal of $A \otimes_P A$ which is generated by the elements of the form $\{1 \otimes x - x \otimes 1 : x \in A\}$ as an A-module. We define a map $T_{A/P}: A \to I_P(A)$ by $T_{A/P}(x) = 1 \otimes x - x \otimes 1$.

Definition 1. Let P be a ring and let A be a P-algebra. Let E be an A-algebra. A map $\tau: A \to E$ is called a *general P-Taylor series* if the following conditions are satisfied :

(1) τ is P-linear,

(2) $\tau(a) = 0$ for all $a \in P$,

(3) if $x, y \in A$, then $\tau(xy) = x\tau(y) + y\tau(x) + \tau(x)\tau(y)$.

Lemma 2. *Let the notation be as above. Then we have :*

(1) *$T_{A/P}$ is a general P-Taylor series.*

(2) *For any generated P-Taylor series τ of A into any A-algebra E, there exists a unique A-algebra map $\rho: I_P(A) \to E$ such that $\tau = \rho \cdot T_{A/P}$.*

Proof. See [4].

Let E be a A-algebra. We say that E is *n-trancated* if, for every sequence $x_0, x_1, \cdots, x_n$ of $n+1$ elements of E, the product $x_0 x_1 \cdots x_n = 0$.

A general P-Taylor series $\tau: A \to E$ is said to be *n-truncated* if E is n-truncated. A 1-truncated general P-Taylor series is called a derivation.

We denote by $I^{n+1}_P(A)$ the $n+1$-power of the ideal $I_P(A)$ and denote by $D^n_P(A)$ the A-algebra $I_P(A)/I^{n+1}_P(A)$. Let $T^n_{A/P}$ be the map $A \to D^n_P(A)$ which is obtained by composing $T_{A/P}$ with the canonical homomorphism $I_P(A) \to D^n_P(A)$. It is easy to see that $T^n_{A/P}$ is an n-truncated general P-Taylor series.

Let P be a topological ring and let A be a topological P-algebra. Then we say that A is *preadmissible,* if there exists an open ideal $\mathfrak{u}$ of A such that for every open neighborhoods $\mathfrak{v}$ of (0) in A there exists an positive an integer m such that $\mathfrak{u}^m \subset \mathfrak{v}$.

PROPOSITION 3. *Let P be a topological ring and let A be a formally smooth P-algebra. Assume that A is a preadmissible ring and that the $n+1$-th powers of all open ideals are open in A. Then $D^n_P(A)$ is a formally projective A-module.*

Proof. See Theorem A2.2 in [1].

It is easy to see that a local ring R with a maximal ideal $\mathfrak{m}$ is preadmissible in $\mathfrak{m}$-adic topology and that if $\mathfrak{a}$ is an open ideal of R then, for any positive integer n, the $n+1$-power of $\mathfrak{a}$ is open in R.

Therefore the following Corollary follows from Proposition 3.

COROLLARY 4. *Let k be a field. Let R be a local ring containing k. Assume that R is a formally smooth over k. Then $D^n_k(R)$ is a formally projective R-module.*

2. Characterizations of regular local rings.

DEFINITION 5. Let P be a topological ring and let A be a topological P-algebra. Let $\mathfrak{a}$ be an ideal of A. $\mathfrak{a}$ is called a *P-differential ideal,* if $\partial\mathfrak{a} \subset \mathfrak{a}$ for any every continuous P-derivation ∂ of A to A.

THEOREM 6. *Let P be a topological ring. Let A be a topological P-algebra and let $\mathfrak{m}$ be an ideal of A. Let $\mathfrak{a}$ be a closed ideal of A. Assume that $D^n_P(A)$ is a formally projective A-module. If it holds that*

$$T^n_{A/P}\,\mathfrak{a} \subset \bigcap_{r=1}^{\infty}(\mathfrak{a}D^n_P(A)+\mathfrak{m}^r D^n_P(A)),$$

$\mathfrak{a}$ is a P-differential ideal. Conversely, assume that A is complete in the $\mathfrak{m}$-adic topology. Then if $\mathfrak{a}$ is a P-differential ideal, we have

$$T^n_{A/P}\,\mathfrak{a} \subset \bigcap_{r=1}^{\infty}(\mathfrak{a}D^n_P(A)+\mathfrak{m}^r D^n_P(A)).$$

Proof. Let $\partial = f\cdot T^n_{A/P}$ be a derivation of A to A, where f is an A-homomorphism $D^n_P(A)\to A$ which is obtained by composing the canonical A-homomorphism $D^n_P(A) \to D^1_P(A)$ with A-homomorphism $D^1_P(A) \to A$. By our assumption, $\partial\mathfrak{a}$ is contained in $\mathfrak{a}+\mathfrak{m}^r$ for every $r=1, 2, \cdots$. Since $\mathfrak{a}$ is a closed ideal, it holds that $\partial\mathfrak{a}\subset\mathfrak{a}$.

Conversely assume that it

$$T^n_{A/P}\,\mathfrak{a} \not\subset \mathfrak{a}D^n_P(A)+\mathfrak{m}^s D^n_P(A)$$

for some positive integer s. Then $T^n_{A/P}\,\mathfrak{a} \bmod \mathfrak{m}^s D^n_P(A)$ is not contained in $\mathfrak{a}(D^n_P(A)/\mathfrak{m}^s D^n_P(A))$. Since $D^n_P(A)$ is a formally projective A-module, $D^n_P(A)/\mathfrak{m}^s D^n_P(A)$ is a projective $A/\mathfrak{m}^s$-module. Therefore $D^n_P(A)/\mathfrak{m}^s D^n_P(A)$ is a direct summand of a free $A/\mathfrak{m}^s$-module F and $T^n_{A/P}\,\mathfrak{a} \bmod \mathfrak{m}^s D^n_P(A)$ is not contained in $\mathfrak{a}F$. Hence there exists an $A/\mathfrak{m}^s$-homomorphism $\psi' : F\to A/\mathfrak{m}^s$ such that $\psi'(T^n_{A/P}\,\mathfrak{a} \bmod \mathfrak{m}^s D^n_P(A))$ is not contained in $\mathfrak{a}+\mathfrak{m}^s/\mathfrak{m}^s$. Let ψ be the restriction of ψ' to $D^n_P(A)/\mathfrak{m}^s D^n_P(A)$. Since A is complete, there exists an A-homomorphism $\varphi : D^n_P(A) \to A$ which induce ψ', by Proposition 5.3 in [1]. By our assumption, $\varphi\cdot T^n_{A/P}\,\mathfrak{a}$ is contained in $\mathfrak{a}$ and hence $\psi'(T^n_{A/P}\,\mathfrak{a} \bmod \mathfrak{m}^s D^n_P(A))$ is contained in $\mathfrak{a}+\mathfrak{m}^s/\mathfrak{m}^s$. But this is a contradiction.

COROLLARY 7. *Let P be a topological ring and let A be a topological P-algebra. Let $\mathfrak{a}$ be a closed ideal of A and put $A'=A/\mathfrak{a}$. Assume that $D^n_P(A)$ is a formally projective A-module. If a continuous A'-homomorphism $\varphi^n_{A'/A/P} : A'\otimes_A D^n_P(A) \to D^n_P(A')$ is formally bimorphic, then $\mathfrak{a}$ is a P-differential ideal. Conversely, assume that A is complete in the $\mathfrak{m}$-adic topology. Then, if $\mathfrak{a}$ is a P-differential ideal, a continuous A'-homomorphism $\varphi^n_{A'/A/P} : A'\otimes_A D^n_P(A)\to D^n_P(A')$ is formally bimorphic.*

Proof. Now consider the following commutative diagrams for all $r=1, 2, \cdots$:

$$\begin{array}{ccccccc}
\mathfrak{a}/\mathfrak{a}^2 & \longrightarrow & A'\otimes_A D^n_P(A) & \xrightarrow{\varphi^n_{A'/A/P}} & D^n_P(A') & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & A'\otimes_A D^n_P(A)/\mathfrak{m}'^r(A'\otimes_A D^n_P(A)) & \xrightarrow{\nu(r)} & D^n_P(A')/\mathfrak{m}'^r D^n_P(A') & &
\end{array}$$

where $\mathfrak{m}'=\mathfrak{m}+\mathfrak{a}/\mathfrak{a}$. Then if $\varphi^n_{A'/A/P}$ is formally bimorphic, we have

$$\varprojlim A'\otimes_A D^n_P(A)/\mathfrak{m}'^r(A'\otimes_A D^n_P(A)) \cong \varprojlim D^n_P(A')/\mathfrak{m}'^r D^n_P(A')$$

by Corollary A2.2 in [1]. Hence we have

$$A'\otimes_A D^n_P(A)/\mathfrak{m}'^r(A'\otimes_A D^n_P(A)) \cong D^n_P(A')/\mathfrak{m}'^r D^n_P(A')$$

for all $r=1, 2, \cdots$. Since $\mathfrak{a}$ is mapped to zero under the homomorphism $A \to A'\otimes_A D_P^n(A)/\mathfrak{m}'^r(A'\otimes_A D_P^n(A))$ and we have

$$A'\otimes_A D_P^n(A)/\mathfrak{m}'^r(A'\otimes_A D_P^n(A)) \cong D_P^n(A)/\mathfrak{a} D_P^n(A)+\mathfrak{m}^r D_P^n(A),$$

hence we have

$$T_{A/P}^n\, \mathfrak{a} \subset \mathfrak{a} D_P^n(A)+\mathfrak{m}^r D_P^n(A)$$

for all $r=1, 2, \cdots$. Therefore $\mathfrak{a}$ is a P-differential ideal by Theorem 6.

Conversely, assume that $\mathfrak{a}$ is a P-differential ideal. Then we have

$$T_{A/P}^n\, \mathfrak{a} \subset \mathfrak{a} D_P^n(A)+\mathfrak{m}^r D_P^n(A)$$

for all $r=1, 2, \cdots$, by Theorem 6. Hence the surjective homomorphism

$$\nu(r)\ :\ A'\otimes_A D_P^n(A)/\mathfrak{m}'^r(A'\otimes_A D_P^n(A)) \to D_P^n(A)/\mathfrak{m}'^r D_P^n(A')$$

is also injective. Thus we have

$$A'\otimes_A D_P^n(A)/\mathfrak{m}'^r(A'\otimes_A D_P^n(A)) \cong D_P^n(A')/\mathfrak{m}'^r D_P^n(A')$$

for all $r=1, 2, \cdots$. Therefore we have

$$\varprojlim A'\otimes_A D_P^n(A)/\mathfrak{m}'^r(A'\otimes_A D_P^n(A)) \cong \varprojlim D_P^n(A')/\mathfrak{m}'^r D_P^n(A').$$

This means that $\varphi_{A'/A/P}^n$ is formally bimorphic.

THEOREM 8. *Let $(R, \mathfrak{m})$ be a noetherian complete local ring containing a field k and let K be a coefficient field of R. Assume that K is separable over k. Then if there exists a complete regular local ring R^* containing K with maximal ideal $\mathfrak{m}^*$ and a K-differential ideal $\mathfrak{a}$ of R^* such that R is K-isomorphic to $R^*/\mathfrak{a}$, $D_K^n(R)$ is a formally projective R-module.*

Proof. Since $R^*/\mathfrak{m}^*$ is separable over k, if $R^*$is regular, $R^*$is formally smooth over k. So $R^*$is formally smooth over K by Lemma (28.0) in [5], since $R^*\otimes_k K \cong R^*$. Then $D_K^n(R^*)$ is a formally projective R^*-module by Corollary 4 and $R\otimes_{R^*} D_K^n(R^*)$ is a formally projective R-module. Since $\mathfrak{a}$ is a K-differential ideal, $D_K^n(R)$ is formally bimorphic to $R\otimes_{R^*} D_K^n(R^*)$ by Corollary 7. Therefore $D_K^n(R)$ is a formally projective R-module by Lemma A 2.5 in [1].

PROPOSITION 9. *Let K be a field of characteristic 0 or a perfect field of characteristic $p \neq 0$. Let $R=K[[x_1, \cdots, x_m]]$ be a formal power series ring. Then there is no proper K-differential ideals in R except (0).*

Proof. In case K is a field of characteristic 0, our proposition is an analogy of Proposition 13.1 in [1]. In case K is a perfect field of characteristic $p \neq 0$, let $\mathfrak{a}$ be a proper K-differential ideal of R. Assume that $\mathfrak{a} \neq (0)$. Then $\mathfrak{a}$ is generated by elements in $K^p[[x_1^p, \cdots, x_m^p]]$ by Proposition 13 in [1]. Therefore we choose an element α of the least leading degree in $\mathfrak{a}$ from $K^p[[x_1^p, \cdots, x_m^p]]$. By our assumption $\mathfrak{a}$ is the same as its radical. Hence we have $\alpha^{\frac{1}{p}} \in \mathfrak{a}$. This is a contradiction, because degree $\alpha^{\frac{1}{p}} <$ degree α.

DEFINITION 10. Let P be a ring and let A be a P-algebra. The algebra of n-order $\mathfrak{m}$-adic differentials of A over P, denoted $\tilde{D}_P^n(A)$, is defined by

$$\tilde{D}_P^n(A)=D_P^n(A)/\bigcap_{r=1}^{\infty} \mathfrak{m}^r D_P^n(A).$$

We denote by $\tilde{T}_{A/P}^n$ the general P-Taylor series $A \to \tilde{D}_P^n(A)$ which is obtained by composing $T_{A/P}^n$ with the canonical homomorphism $D_P^n(A) \to \tilde{D}_P^n(A)$. It follows from the above remarks that, for any general P-Taylor series τ of A into any separated n-truncated A-algebra E, there exist a unique A-algebra homomorphism $\rho : \tilde{D}_P^n(A) \to E$ such that $\tau=\rho\cdot\tilde{T}_{A/P}^n$.

From now on we will assume that R is a reduced noetherian local ring. Assume also that the form ring of R is reduced.

THEOREM 11. *Let $(R, \mathfrak{m})$ be a noetherian local ring with coefficient field K of characteristic 0. Then R is a regular local ring if and only if $D_K^n(R)$ is a formally projective R-module.*

Proof. We may assume that R is $\mathfrak{m}$-adic complete by Corollary A2.2 and Lemma A2.5 in [1]. We first show the only if part. If R is a complete regular local ring, we may express R as a formal power series ring $K[[x_1, \cdots, x_m]]$.

So $D_K^n(R)$ is a formally projective R-module by Proposition 9 and Theorem 8.

Next we show the if part. Assume that $D_K^n(R)$ is a formally projective R-module. Then $\tilde{D}_K^n(R)$ is an $\mathfrak{m}$-adic free R-module by Corollary 1 in [2]. Since R is a noetherian complete local ring, we may express

$$R \cong K[[x_1, \cdots, x_m]]/\mathfrak{b}$$

for some ideal $\mathfrak{b} \subset K[[x_1, \cdots, x_m]]$. We put $R^* = K[[x_1, \cdots, x_m]]$. Then $\tilde{D}_K^n(R^*)$ is a finitely generated R^*-module by Theorem 1.12 in [3]. Since we have

$$\tilde{D}_K^n(R) \cong R \otimes_{R^*} \tilde{D}_K^n(R^*) / R \otimes_{R^*} R^* \tilde{T}_{R^*/K}^n(\mathfrak{b}).$$

$\tilde{D}_K^n(R)$ is also a finitely generated R-module. Hence $\tilde{D}_K^n(R)$ is a free R-module by Theorem 5.1 in [1]. Thus $\tilde{D}_K^n(R)$ is a finitely generated free R-module. Therefore R is a regular local ring by Theorem 2.3 in [3].

THEOREM 12. *Let $(R, \mathfrak{m})$ be a noetherian local ring with coefficient field K of characteristic $p \neq 0$. Assume that K is perfect. Then if R is a regular local ring, $D_K^n(R)$ is formally projective R-module. Conversely, if R is analytically unramified and $D_K^n(R)$ is a formally projective R-module, R is a regular local ring.*

Proof. The assertion can be proved in a similar way as in the proof of Theorem 11.

Since, for a finitely generated module over local ring, the flatness and the freedom come to the same thing, the following Corollary follow from Theorem 2.3 in [3].

COROLLARY 13. *Let $(R, \mathfrak{m})$ be a noetherian local ring with coefficient field K. Assume that $D_K^n(R)$ is a finitely generated R-module. Then R is a regular local ring if and only if $D_K^n(R)$ is flat.*

COROLLARY 14. *Let $(R, \mathfrak{m})$ be a noetherian complete local ring with coefficient field K. Then R is a regular local ring if and only if $\tilde{D}_K^n(R)$ is flat.*

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(Received January 25, 1986)