A Remark on the n-valued Algebroid Functions

Yûji HASHIMOTO

n 価代数型函数についての 1 注意

橋　本　有　司

In this paper, we show some property of the n-valued algebroid functions of $\lambda = n-1$, and then give proofs of some well-known theorems concerning the deficiencies of n-valued entire algebroid functions and the Picard constants of n-sheeted regularly branched covering surfaces of $|z| < \infty$.

§ 1. Introduction.

Let $w(z)$ be an n-valued transcendental algebroid function with the irreducible defining equation $f_0(z)w^n + f_1(z)w^{n-1} + \cdots\cdots + f_n(z) = 0$, and λ the maximum number of $\mathbb{C}$-independent linear relations among the coefficients $f_0(z), f_1(z), \cdots\cdots, f_n(z)$. In this paper, in §2, we shall show some property of $w(z)$ for $\lambda = n-1$, and then, in §3, give proofs of some theorems, though they are not new, in the theory of algebroid functions.

We shall use the standard symbols of the Nevanlinna theory of algebroid functions and systems (see Selberg 〔6〕 and Cartan 〔2〕).

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§ 2. Now, we shall give the following theorem.

Theorem. Let $w(z)$ be as in §1. If $\lambda = n-1$, then,
(1) there exist an algebraic function $\tilde{w}(\zeta)$ (the defining equation is $F(\zeta, \tilde{w}) = \tilde{w}^n + p_1(\zeta)\tilde{w}^{n-1} + \cdots\cdots + p_n(\zeta) = 0$, where $p_1(\zeta), \cdots\cdots, p_n(\zeta)$ are polynomials of degree at most 1 of ζ) and a meromorphic function $\zeta = \varphi(z)$, and $w(z)$ is represented by $\tilde{w}(\varphi(z))$. Further,
(2) we have $T(r,w) = \frac{1}{n}T(r,\varphi) + O(1)$ and $N(r,w_0,w) = \frac{1}{n}N(r, \zeta_0, \varphi)$ (therefore, $\delta(w_0,w) = \delta(\zeta_0,\varphi)$), where ζ_0 is determined by the equation $F(\zeta_0, w_0) = 0$.

Proof. (1) As $\lambda = n-1$, there exist two linearly independent functions among $f_0(z), f_1(z), \cdots\cdots, f_n(z)$ (we assume them to be $f_0(z)$ and $f_1(z)$), and the other remaining $n-1$ functions are represented by the linear combinations of them. So that, the defining equation of $w(z)$ is $f_0(z)w^n + f_1(z)w^{n-1} + (\alpha_2 f_0(z) + \beta_2 f_1(z))w^{n-2} + \cdots\cdots + (\alpha_n f_0(z) + \beta_n f_1(z)) = 0$, and setting $\zeta = \varphi(z) = \frac{f_1(z)}{f_0(z)}$, we obtain (1).

(2) For the proof of $T(r,w) = \frac{1}{n}T(r,\varphi) + O(1)$, we consider the system $f(z) = (f_0(z), f_1(z), \cdots\cdots, f_n(z))$. Then, $T(r,w) = \frac{1}{n}T(r,f) + O(1)$, so that, we may show $T(r,f) = T(r,\varphi) + O(1)$. $T(r,\varphi) \leqq T(r,f) + O(1)$ is well-known. For the opposite inequality, we have

$$T(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log \max\{|f_0(z)|, |f_1(z)|, |\alpha_2 f_0(z) + \beta_2 f_1(z)|, \cdots, |\alpha_n f_0(z) + \beta_n f_1(z)|\} d\theta \quad (z = re^{i\theta})$$

$$\leqq \frac{1}{2\pi}\int_0^{2\pi} \log K \max\{|f_0(z)|, |f_1(z)|\} d\theta \quad (K = \max\{1, |\alpha_2| + |\beta_2|, \cdots, |\alpha_n| + |\beta_n|\})$$

$$= T(r,\varphi) + O(1).$$

For the proof of $N(r,w_0,w) = \frac{1}{n}N(r,\zeta_0,\varphi)$, let z_0 be a point at which $\tilde{w}(\zeta_0) = w_0$ and $\varphi(z_0) = \zeta_0$. By expanding $\tilde{w}(\zeta)$ at ζ_0, we have $\tilde{w} - w_0 = a_1(\zeta - \zeta_0)^{\frac{1}{m_0}} + \cdots\cdots$ ($a_1 \neq 0$), for, the neighborhoods of w_0 and ζ_0 are corresponding 1 to m_0. So that, if $\zeta - \zeta_0 = b_{k_0}(z - z_0)^{k_0} + \cdots\cdots$ ($b_{k_0} \neq 0$), we have $\tilde{w} - w_0 = a_1 b_{k_0}(z - z_0)^{\frac{k_0}{m_0}} + \cdots\cdots$, that is, the multiplicities of w_0 and ζ_0 at z_0 are the same. Q. E. D.

§ 3. Applying this theorem, we shall give proofs of the following theorems.

Theorem A (Niino-Ozawa 〔4〕 and Toda 〔7〕). Let $w(z)$ be an n-valued transcendental entire algebroid function with the deficiency condition

$$\sum_{j=1}^{2n-1} \delta(w_j, w) > 2n-2$$

for some $\{w_j\}_{j=1}^{2n-1}$ not including ∞, then at least $n-1$ values of $\{w_j\}_{j=1}^{2n-1}$ are Picard exceptional values of $w(z)$.

Proof. According to Toda 〔7〕, the deficiency condition implies $\lambda = n-1$, so that, $w(z)$ has the property mentioned in §2. Now, let S be the Riemann surface defined by the algebraic function $\tilde{w}(\zeta)$, then, as $\zeta = \varphi(z)$ $(= f_1(z))$ is entire, $w(z)$ does not take the values over $\zeta = \infty$ on S. We shall show that there are no branch points over $\zeta = \infty$ on S. Suppose there exist some branch points over $\zeta = \infty$. Then, we have

$$\delta(\infty,w)+\sum_{j=1}^{2n-1}\delta(w_j,w)=\delta(\infty,\varphi)+\sum_{j=1}^{2n-1}\delta(\zeta_j,\varphi),$$

where ζ_j is determined by the equation $F(\zeta_j,w_j)=0$. Here, as there are at most $n-1$ points over $\zeta=\infty$ on S, we have

$$\delta(\infty,\varphi)+\sum_{j=1}^{2n-1}\delta(\zeta_j,\varphi)$$

$$\leqq (n-1)\delta(\infty,\varphi)+n\sum_{\zeta\neq\infty}\delta(\zeta,\varphi)$$

$$\leqq (n-1)+n=2n-1.$$

This is a contradiction, and we see that there are no branch points over $\zeta=\infty$, that is, there are at least $n-1$ Picard values except for ∞. Q. E. D.

According to the proof above, we see that the essential part is given by Toda [7] concerning the Cartan's conjecture, and, at that part, the quantitative condition implies the qualitive conclusion.

The following theorem can also be given along the same line.

Theorem B (Ozawa [5] and Aogai [1]). Let R be an n-sheeted regularly branched covering surface of $|z|<\infty$ and P(R) the Picard constant of R. If $P(R)>\frac{3}{2}n$, then $P(R)=2n$ and the defining equation of R is, after linear fractional change of the coordinates z and w, $w^n=(e^{H(z)}-A)(e^{H(z)}-B)^{n-1}$, where H(z) is entire and A and B are constants ($A\neq 0$ and $B\neq 0$).

Proof. Let w(z) be a meromorphic function on R with k ($>\frac{3}{2}n$) Picard exceptional values. According to Dufresnoy [3] , we have $\lambda=n-1$ for w(z), so that, w(z) has the property mentioned in §**2**. Now, let S be the Riemann surface defined by the algebraic function $\tilde{w}(\zeta)$, then, as R is regularly branched, S is also regularly branched. Further, being of of genus 0, by Hurwitz's formula, S has only two branch points of order $n-1$. Therefore, after linear fractional change of coordinates, we have $w^n=K\zeta$. And, as w(z) has k ($>\frac{3}{2}n\geqq n+1$) Picard values, $\zeta=\varphi(z)$ has two Picard values A and B ($A\neq 0$ and $B\neq 0$), that is, $\zeta=B\frac{e^{H(z)}-A}{e^{H(z)}-B}$. Thus, we obtain the theorem. Q. E. D.

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