Remarks on $\mathfrak{m}$-adic Higher Differential Theoretic Characterization of Regular Local Rings

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Abstract. This is a suite of the previous paper [1]. In that paper, we showed under some assumptions that when R is a local ring of equal characteristic with maximal ideal $\mathfrak{m}$, then the algebra $\widehat{D}_N(R,P)$ of $\mathfrak{m}$-adic P-differentials of rank N ($\neq \mathbf{N}$) in R is free if and only if R is regular. In this paper, we shall show under some assumptions that when R is a local ring of unequal characteristic with maximal ideal $\mathfrak{m}$, then $\widehat{D}_N(R,P)$ is free if and only if R is regular and $\mathfrak{m}^2$ does not contain a prime element u of P.

§ 1. Preliminaries.

Throughout this paper, all rings will be assumed to be commutative rings with unit elements. Let P be a ring and let R be a P-algebra. Let N be a set $\{1,2,\cdots,n\}$ or the set $\mathbf{N}$ of natural numbers and let N_0 be $N\cup\{0\}$. The algebra of P-differentials of rank N in R will be denoted by $\widehat{D}_N(R,P)$ and associated universal P-derivation of rank N from R into $\widehat{D}_N(R,P)$ will be denoted by $\mathbf{d}_N=\{d^i_{R,P}\}_{i\in N_0}$. Furthermore we shall assume that R is an $\mathfrak{m}$-adic ring. Then the algebra of $\mathfrak{m}$-adic P-differentials of rank N in R will be denoted by $\widehat{D}_N(R,P)$ and associated universal P-derivation of rank N from R into $\widehat{D}_N(R,P)$ will be denoted by $\widehat{\mathbf{d}}_N=\{\widehat{d}^i_{R,P}\}_{i\in N_0}$. In this paper, when we call R an $\mathfrak{m}$-adic ring, we always assume that an ideal $\mathfrak{m}$ of R satisfies the condition $\bigcap_{r\geq 1}\mathfrak{m}^r=0$.

§ 2. Characterizations of regular local rings. (unequal characteristic case)

Let R be a P-algebra and let $\mathfrak{P}$ be a prime ideal of R and let $\mathfrak{p}$ be the contraction of $\mathfrak{P}$ in P. Then we shall say tnat $\mathfrak{P}$ is unramified if the following conditions are satisfied:

(1) $\mathfrak{p}R_{\mathfrak{P}}=\mathfrak{P}R_{\mathfrak{P}}$,

(2) $R_{\mathfrak{P}}/\mathfrak{P}R_{\mathfrak{P}}$ is a finite separable extention of $P_{\mathfrak{p}}/\mathfrak{p}P_{\mathfrak{p}}$.

Let $\mathfrak{p}$ be a prime ideal of P, then R will be said to be unramified over $\mathfrak{p}$ if the following conditions are satisfied:

(1′) every prime ideal $\mathfrak{P}$ of R such that $\mathfrak{P}\cap P=\mathfrak{p}$ is unramified,

(2′) there exist only a finite number of primes in R such that $\mathfrak{P}\cap P=\mathfrak{p}$.

We shall say that R is unramified over P if R is unramified over every prime ideal $\mathfrak{p}$ of P.

PROPOSITION 1. *Let R be a P-algebra and let $\mathfrak{P}$ be a prime ideal of R. Let $\mathfrak{p}$ be the contraction of $\mathfrak{P}$ in P. Then if $\mathfrak{P}$ is unramified, it holds that $\widehat{D}_N(R_{\mathfrak{P}},P_{\mathfrak{p}})=0$. Conversely,* *if R is noetherian and $R_{\mathfrak{P}}/\mathfrak{P}R_{\mathfrak{P}}$ is finitely generated over $P_{\mathfrak{p}}/\mathfrak{p}P_{\mathfrak{p}}$, then from $\widehat{D}_N(R_{\mathfrak{P}},P_{\mathfrak{p}})=o$ $\mathfrak{P}$ is unramified.*

PROOF. We denote by k and K the fields $P_{\mathfrak{p}}/\mathfrak{p}P_{\mathfrak{p}}$ and $R_{\mathfrak{P}}/\mathfrak{P}R_{\mathfrak{P}}$ respectively. Now assume that $\mathfrak{P}$ is unramified. Since K is a finite separable extension of k, we have, by Cor. 2 of Prop. 1.6 in [1], $\widehat{D}_N(K,k)=\widehat{D}_N(K,P_{\mathfrak{p}})=D_N(K,P_{\mathfrak{p}})=0$. Hence $\widehat{D}_N(K,P_{\mathfrak{p}})_i=0$ for each $i\in N$ where $\widehat{D}_N(K,P_{\mathfrak{p}})_i$ is the K-submodule of $\widehat{D}_N(K,P_{\mathfrak{p}})$. By the exact sequence

$$\mathfrak{P}R_{\mathfrak{P}}\ /\ (\mathfrak{P}R_{\mathfrak{P}})^2\longrightarrow \widehat{D}_N(R_{\mathfrak{P}},P_{\mathfrak{p}})_i\ /\ \widehat{I}_N(\mathfrak{P}R_{\mathfrak{P}})_i\longrightarrow \widehat{D}_N(K,P_{\mathfrak{p}})_i=0$$

for each $i\in N$ where $\widehat{I}_N(\mathfrak{P}R_{\mathfrak{P}})_i$ is the submodule of $\widehat{D}_N(R_{\mathfrak{P}},P_{\mathfrak{p}})_i$ generated by all elements of the form $w_s\,\hat{d}^{i-s}x$ such that $x\in\mathfrak{P}R_{\mathfrak{P}}$, $w_s\in\widehat{D}_N(R,P)_s$ and $s=1,\cdots,i$ for $i>o$, $\widehat{D}_N(R_{\mathfrak{P}},P_{\mathfrak{p}})_i$ is generated by the elements $\hat{d}^{s_1}_{R_{\mathfrak{P}},P_{\mathfrak{p}}}x_1\cdots\hat{d}^{s_r}_{R_{\mathfrak{P}},P_{\mathfrak{p}}}x_r$'s where $x_1,\cdots,x_r$ are any set of generators of $\mathfrak{P}R_{\mathfrak{P}}$ and $s_1+\cdots+s_r=i$, $r\geqq 2$. Since $\mathfrak{P}R_{\mathfrak{P}}$ is generated by the elements in $\mathfrak{p}R_{\mathfrak{P}}$, we see that $\widehat{D}_N(R_{\mathfrak{P}},P_{\mathfrak{p}})_i=0$ for each $i\in N$. Therefor we have $\widehat{D}_N(R_{\mathfrak{P}},P_{\mathfrak{p}})=0$.

Conversely assume that $\widehat{D}_N(R_{\mathfrak{P}},P_{\mathfrak{p}})=0$, then we have $\widehat{D}_N(R_{\mathfrak{P}},P_{\mathfrak{p}})_1=0$. Hence it follows, with the same reasoning as in Lemma 4 of Part 2 in [6], that $D_N(K,k)_1=0$. This means that K is separably algebraic over k. Therefore $R_{\mathfrak{P}}\ /\ \mathfrak{p}R_{\mathfrak{P}}$ is a local ring with the maximal ideal $\mathfrak{P}R_{\mathfrak{P}}/\mathfrak{p}R_{\mathfrak{P}}$ and $R_{\mathfrak{P}}/\mathfrak{p}R_{\mathfrak{P}}$ contains the field k such that K is separably algebraic over k. Hence we can use Lemma 4′ of Part 2 in [6] and we get $\mathfrak{P}R_{\mathfrak{P}}/\mathfrak{p}R_{\mathfrak{P}}=(\mathfrak{P}R_{\mathfrak{P}}/\mathfrak{p}R_{\mathfrak{P}})^2$. Since, by our assumptions, $\mathfrak{P}R_{\mathfrak{P}}/\mathfrak{p}R_{\mathfrak{P}}$ has a finite set of generators, we must have $\mathfrak{P}R_{\mathfrak{P}}=\mathfrak{p}R_{\mathfrak{P}}$.

PROPOSITION 2. *Let R be an $\mathfrak{m}$-adic P-algebra and let $\mathfrak{P}$ be a prime ideal of R containing $\mathfrak{m}$. Assume that $\widehat{D}_N(R,P)$ is a finite R-algebra for $N\neq\mathbf{N}$. Let $\mathfrak{A}_i$ be the annihilator of $\widehat{D}_N(R,P)_i$ in R. Then if $\mathfrak{P}$ is unramified, $\mathfrak{P}$ does not contain $\mathfrak{A}_i$ for each $i\in N$.*

PROOF. By our assumptions, we see that $R_{\mathfrak{P}}$ is unramified over $P_{\mathfrak{p}}$. Hence $R_{\mathfrak{P}}/\mathfrak{P}R_{\mathfrak{P}}$ is a finite separable extension of $P_{\mathfrak{p}}/\mathfrak{p}P_{\mathfrak{p}}$, thus we have $D_N(R_{\mathfrak{P}}/\mathfrak{P}R_{\mathfrak{P}},P_{\mathfrak{p}})=0$ by Cor. 2 of Prop. 9 in [3]. Moreover $\widehat{D}_N(R_{\mathfrak{P}}/\mathfrak{P}R_{\mathfrak{P}},P_{\mathfrak{p}})=D_N(R_{\mathfrak{P}}/\mathfrak{P}R_{\mathfrak{P}},P_{\mathfrak{p}})$ by Cor. 2 of Prop. 1. 6 in [1], hence $\widehat{D}_N(R_{\mathfrak{P}}/\mathfrak{P}R_{\mathfrak{P}},P_{\mathfrak{p}})=0$. From this $\widehat{D}_N(R_{\mathfrak{P}}/\mathfrak{P}R_{\mathfrak{P}},P_{\mathfrak{p}})_i=0$ for each $i\in N$. Then, by the exact sequence

$$\mathfrak{P}R_{\mathfrak{P}}/(\mathfrak{P}R_{\mathfrak{P}})^2\longrightarrow\widehat{D}_N(R_{\mathfrak{P}},P_{\mathfrak{p}})_i\ /\ \widehat{I}_N(\mathfrak{P}R_{\mathfrak{P}})_i\longrightarrow \widehat{D}_N(R_{\mathfrak{P}}/\mathfrak{P}R_{\mathfrak{P}},P_{\mathfrak{p}})_i=0$$

for each $i\in N$, $\widehat{D}_N(R_{\mathfrak{P}},P_{\mathfrak{p}})_i$ is generated by the form $\hat{d}^{s_1}_{R_{\mathfrak{P}},P_{\mathfrak{p}}}y_1\cdots\hat{d}^{s_r}_{R_{\mathfrak{P}},P_{\mathfrak{p}}}y_r$ where $y_1,\cdots,y_r$ are any set of generators of $\mathfrak{P}R_{\mathfrak{P}}$ and $s_1+\cdots+s_r=i$, $r\geqq 2$. Since $\mathfrak{P}R_{\mathfrak{P}}$ is generated by the elements in $\mathfrak{p}P_{\mathfrak{p}}$, we see that $\widehat{D}_N(R_{\mathfrak{P}},P_{\mathfrak{p}})_i=0$ for each $i\in N$. Since, $\widehat{D}_N(R,P)$ is a finite R-algebra, $R_{\mathfrak{P}}\otimes_R\widehat{D}_N(R,P)$ is a finite $R_{\mathfrak{P}}$-algebra. Therefore $R_{\mathfrak{P}}\otimes_R\widehat{D}_N(R,P)$ is Hausdorff from Lemma 1.1 in [1]. From this and the fact that $\widehat{D}_N(R_{\mathfrak{P}},P)\cong R_{\mathfrak{P}}\otimes_R D_N(R,P)$, we can obtain $\widehat{D}_N(R_{\mathfrak{P}},P)\cong R_{\mathfrak{P}}\otimes_R\widehat{D}_N(R,P)$. Hence $\widehat{D}_N(R_{\mathfrak{P}},P)_i\cong R_{\mathfrak{P}}\otimes_R\widehat{D}_N(R,P)_i$ for each $i\in N$ and the annihilator of $\widehat{D}_N(R_{\mathfrak{P}},P)_i$ is given by $\mathfrak{A}_i\otimes_R R_{\mathfrak{P}}$ for each $i\in N$. The above results implies that the annihilator $\mathfrak{A}_i\otimes_R R_{\mathfrak{P}}$ of $\widehat{D}_N(R_{\mathfrak{P}},P)_i$ must be a unit ideal for each $i \in N$, hence $\mathfrak{P}$ cannot contain the annihilator $\mathfrak{A}_i$ for each $i \in N$.

COROLLARY 1. *Let R be a noetherian $\mathfrak{m}$-adic P-algebra. Let $\mathfrak{A}_i$ be the annihilator of $\widehat{D}_N(R,P)_i$ in R. Assume that $R/\mathfrak{m}$ is finitely generated over $P/P \cap \mathfrak{m}$ and $\widehat{D}_N(R,P)$ is a finite R-algebra for $N \neq \mathbf{N}$. Then a prime ideal $\mathfrak{P}$ in R containing $\mathfrak{m}$ is unramified if and only if $\mathfrak{P}$ does not contain $\mathfrak{A}_i$ for each $i \in N$.*

PROOF. By our assumption and Prop. 1, $\mathfrak{P}$ is unramified if and only if $\widehat{D}_N(R_{\mathfrak{P}},P_{\mathfrak{p}})=0$. Hence $\mathfrak{P}$ is unramified if and only if $\widehat{D}_N(R_{\mathfrak{P}},P_{\mathfrak{p}})_i=0$ for each $i \in N$. Since the annihilator of $\widehat{D}_N(R_{\mathfrak{P}},P_{\mathfrak{p}})_i$ is given by $\mathfrak{A}_i R_{\mathfrak{P}}$ in the same way as in proof of Prop. 2, we have our assertion.

COROLLARY 2. *Let R be an $\mathfrak{m}$-adic P-algebra. Assume that $\widehat{D}_N(R,P)$ is a finite R-algebra for $N \neq \mathbf{N}$. Then if R is unramified over P, $\widehat{D}_N(R,P)=0$.*

PROOF. We assume that $\widehat{D}_N(R,P)_i \neq 0$ for some $i \in N$. Then the annihilator $\mathfrak{A}_i$ of $\widehat{D}_N(R,P)_i$ is not a unit ideal, and there exists a maximal ideal $\mathfrak{n}$ containing $\mathfrak{A}_i$. Hence $\mathfrak{n}$ must be the one which is ramified over P. This is a contradiction.

LEMMA 3. *Let R be a local ring of characteristic 0 with maximal ideal $\mathfrak{m}$ and with a residue field of prime characteristic p. Let P be a discrete valuation ring dominated by R and let u be a prime element of P. Assume that $R/\mathfrak{m}$ is separably algebric over P/uP. Then we have the following*:

(1) *There exists a complete discrete valuation ring P' containing P such that P' has the same prime element u as P and $P'/uP' \cong R/\mathfrak{m}$.*

(2) *Assume that $\widehat{D}_N(R,P)$ and $\widehat{D}_N(R,P')$ are finite R-algebras for $N \neq \mathbf{N}$, we have $\widehat{D}_N(R,P) = \widehat{D}_N(R,P')$ for the valuation ring P' satisfying (1).*

PROOF. The assertion can be proved in a similar way as in the proof of Lemma 6 in [5].

THEOREM 4. *Let R be a local ring of characteristic 0 with maximal ideal $\mathfrak{m}$ and with a residue field of prime characteristic p. Let P be a discrete valuation ring dominated by R and let u be a prime element of P. Assume that $R/\mathfrak{m}$ is finitely generated separable extension of P/uP and $\widehat{D}_N(R,P)$ is a finite R-algebra for $N \neq \mathbf{N}$. Then $\widehat{D}_N(R,P)$* is *$\mathfrak{m}$-adic free algebra if and only if R is a regular local ring and $u \notin \mathfrak{m}^2$.*

PROOF. Let $\alpha_1,\cdots,\alpha_r$ be elements of R such that their residue classes $\bar{\alpha}_1,\cdots,\bar{\alpha}_r$ modulo $\mathfrak{m}$ are separating transcendent base of $R/\mathfrak{m}$ over P/uP. Now assume that R is a regular local ring and $u \notin \mathfrak{m}^2$ and let $u, u_1,\cdots,u_t$ be a minimal basis of $\mathfrak{m}$. Since $\bar{\alpha}_1,\cdots,\bar{\alpha}_r$ are separating transcendent base of $R/\mathfrak{m}$ over P/uP, there exists a discrete valuation ring P_1, dominated by R such that u is a prime element of P_1 and $P_1/uP_1=(P/uP)(\bar{\alpha}_1,\cdots,\bar{\alpha}_r)$. Let R^* be the completion of R and let P' be a complete discrete valuation ring constructed for R^* and P_1

as in Lemma 3. By our assumption, we see that $R^*=P'[[x_1,\cdots,x_t]]$, hence $\widehat{D}_N(R^*,P')$ is a free R-algebra. Since, from (2) in Lemma 3, $\widehat{D}_N(R^*,P_1)=\widehat{D}_N(R^*,P')$, $\widehat{D}_N(R^*,P_1)$ is a free R^*-algebra. On the other hand P_1 is a quotient ring of $P[\alpha_1,\cdots,\alpha_r]$ and $\widehat{D}_N(P[\alpha_1,\cdots,\alpha_r],P)$ is also a free $P[\alpha_1,\cdots,\alpha_r]$-algebra. Hence $\widehat{D}_N(P_1,P)$ is a free P_1-algebra with the base $\{\widehat{d}^{\,i}_{P_1,P}\,\alpha_1,\cdots,\widehat{d}^{\,i}_{P_1,P}\,\alpha_r \mid i\in N\}$. Since $\widehat{D}_N(R,P)$ is a finite R-algebra, $\widehat{D}_N(R^*,P)=R^*\otimes_R\widehat{D}_N(R,P)$ is a finite R^*-algebra and we can obtain the following exact sequence

$$0\longrightarrow R^*\otimes_P\widehat{D}_N(P',P)\longrightarrow\widehat{D}_N(R^*,P)\longrightarrow\widehat{D}_N(R^*,P')\longrightarrow 0$$

from the exact sequence (3) of §1 in [1] by a similar argument as Cor. of Th. 1 in [5]. From the above exact sequence, we see that $\widehat{D}_N(R^*,P)$ is a polynomial ring in variable $\{\widehat{d}^{\,i}_{R^*,P}\,x_1,\cdots,\widehat{d}^{\,i}_{R^*,P}\,x_t,\ \widehat{d}^{\,i}_{R^*,P}\,\alpha_1,\cdots,\widehat{d}^{\,i}_{R^*,P}\,\alpha_r \mid i\in N\}$ since $\widehat{D}_N(P',P)$ is a polynomial ring in variable $\{\widehat{d}^{\,i}_{P',P}\,\alpha_1,\cdots,\widehat{d}^{\,i}_{P',P}\,\alpha_r \mid i\in N\}$. Since we may take x_i's in R such that $u, x_1, \cdots, x_t$ form a regular system of parameters, it follows that $\widehat{D}_N(R,P)$ is a finite R-algebra by corresponding $\widehat{d}^{\,i}_{R^*,P}\,x_j$, $\widehat{d}^{\,i}_{R^*,P}\,\alpha_l$ to $1\otimes\widehat{d}^{\,i}_{R,P}\,x_j$, $1\otimes\widehat{d}^{\,i}_{R,P}\,\alpha_l$ $(j=1,\cdots,t,\ l=1,\cdots,r)$ respectively.

Conversely, assume that $\widehat{D}_N(R,P)$ is a finite R-algebra. Then $\widehat{D}_N(R,P)_1$ is at the same time a finite free R-module. Thus the assertion can be proved in a similar way as Th. 9 in [5].

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