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〈査読付論文〉

The existence of the bilinear forms on the quantum affine superalgebras of type $D^{(1)}(2,1;x)$ $(x \in \mathbb{C} \setminus \{0,-1\})$

$D^{(1)}(2,1;x)$ $(x \in \mathbb{C} \setminus \{0,-1\})$ 型量子アフィン・スーパー代数上の双 1 次形式の存在

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Abstract. We will prove the existence of the bilinear forms on the quantum affine superalgebras of type $D^{(1)}(2,1;x)$ $(x \in \mathbb{C} \setminus \{0,-1\})$.

1 Introduction

In the theory of infinite integrable analysis, the R-matrices play important rules of the integrability of the infinite integrable systems. The R-matrices are generated by the universal R-matrices and the representations of various quantum algebras. In [7], we describe the universal R-matrices of untwisted quantum affine algebras in certain multiplicative formulas by using a new concrete method of constructing all convex orders on the positive root systems. In the work, we use J. Beck's papers [1] and [2] on the Drinfeld second realization of the untwisted quantum affine algebras. On the other hand, in [4], they obtain the Drinfeld second realization of the quantum affine superalgebras of type $D^{(1)}(2,1;x)$, where $x \in \mathbb{C} \setminus \{0,-1\}$.

Our purpose is to extend the results of the paper [7] to quantum affine superalgebras of type $D^{(1)}(2,1;x)$ by using the paper [4]. This paper is the first step toward the aim. To achieve the purpose, it is important to construct the bilinear forms on the quantum superalgebra of type $D^{(1)}(2,1;x)$. In this paper, we prove the existence of the bilinear forms by using a manner similar to Tanisaki's in [15]. We plan that the second step is the construction of convex bases of the quantum superalgebra of type $D^{(1)}(2,1;x)$ by using paper [4] and the third step is the calculation of the values of the bilinear forms on the convex bases.

This paper is organized as follows. In section 2, we recall the notations for the simple root systems of Lie superalgebra of type $D^{(1)}(2,1;x)$. Especially, we give the definition of the inner products on the dual spaces of the Cartan subalgebras. In section 3, we define the quantum affine superalgebras of type $D^{(1)}(2,1;x)$ and give the preliminary results. In section 4, we construct the bilinear forms on the quantum affine superalgebras of type $D^{(1)}(2,1;x)$. Our main result of this paper is Theorem 4.10.

2 Notations for the simple root systems of type $D^{(1)}(2,1;x)$

In this section, we give notations for the simple root systems of Lie superalgebra of type $D^{(1)}(2,1;x)$.

Lie superalgebra $\mathfrak{g}$ is a $\mathbb{Z}_2$-graded algebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ equipped with a super bracket satisfying the super Jacobi identity. The Lie superalgebras, like the Lie algebras, can be studied with the help of Cartan matrices and Dynkin diagrams, but an important difference between Lie algebras and Lie superalgebras is that, in contrast to the Lie algebras, there are several unequivalent simple root systems for each Lie superalgebra with respect to the inner product. Hence, in general, there are several unequivalent Dynkin diagrams for each Lie superalgebra.

Let $\mathfrak{g}$ be the Lie superalgebra of type $D(2,1;x)$, and $\hat{\mathfrak{g}}$ the untwisted affine Lie superalgebra of type $D^{(1)}(2,1;x)$, where $x \in \mathbb{C} \setminus \{0,-1\}$. It is known that there are five unequivalent simple root systems for $\hat{\mathfrak{g}}$ (cf. [4]). So let $\mathcal{D} = \{0,1,2,3,4\}$ be the set of index of Dynkin diagrams of $\hat{\mathfrak{g}}$. For each $d \in \mathcal{D}$, let $\Pi_d = \{\alpha_{i,d} \,|\, i \in \mathrm{I}\}$ be the set of simple roots with $\mathrm{I} = \{0,1,2,3\}$. We define Q_d to be the $\mathbb{Z}$-lattice spanned by Π_d. Then we set $Q_d^+ := \sum_{i\in \mathrm{I}} \mathbb{Z}_{\geq 0}\alpha_{i,d} \subset Q_d$ and $Q_d^- := -Q_d^+$.

For each $d \in \mathcal{D}$, let V_d be a four dimensional $\mathbb{C}$-vector space which spanned by $\Pi_d = \{\alpha_{i,d} \,|\, i \in \mathrm{I}\}$. A symmetric bilinear form $(\,|\,) = (\,|\,)_d : V_d \times V_d \to \mathbb{C}$ is explicitly given as follows:

$$
\begin{aligned}
&(\alpha_{0,0}|\alpha_{0,0}) = 0, && (\alpha_{i,0}|\alpha_{j,0}) = 0 \quad \text{for } i,j \in \mathrm{I}\setminus\{0\}, i \neq j,\\
&(\alpha_{1,0}|\alpha_{1,0}) = -2x, && (\alpha_{1,0}|\alpha_{0,0}) = x,\\
&(\alpha_{2,0}|\alpha_{2,0}) = 2(x+1), && (\alpha_{2,0}|\alpha_{0,0}) = -x-1,\\
&(\alpha_{3,0}|\alpha_{3,0}) = -2, && (\alpha_{3,0}|\alpha_{0,0}) = 1,
\end{aligned}
$$

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and for $d = 1, 2, 3$,

$$(\alpha_{i,d}|\alpha_{j,d}) = (\alpha_{f_d(i),0}|\alpha_{f_d(j),0}),$$

where f_d are the elements of the following subgroup $\mathcal{K}_4$ of the permutation group of I:

$$\mathcal{K}_4 = \{f_0 := \mathrm{id},\ f_1 := (01)(23),\ f_2 := (02)(13),\ f_3 := (03)(12)\},$$

and finally for $d = 4$,

$$\begin{array}{ll} (\alpha_{i,4}|\alpha_{i,4}) = 0 \quad \text{for } i \in \mathrm{I}, & (\alpha_{0,4}|\alpha_{3,4}) = (\alpha_{1,4}|\alpha_{2,4}) = -1, \\ (\alpha_{0,4}|\alpha_{1,4}) = (\alpha_{2,4}|\alpha_{3,4}) = -x, & (\alpha_{0,4}|\alpha_{2,4}) = (\alpha_{1,4}|\alpha_{3,4}) = x + 1. \end{array}$$

For each $d \in \mathcal{D}$ and $\alpha_{i,d} \in \Pi_d$, let $p(\alpha_{i,d})$ be the parity of $\alpha_{i,d}$, which are defined as follows:

$$p(\alpha_{i,d}) = \begin{cases} 0 & \text{if } (\alpha_{i,d}|\alpha_{i,d}) \neq 0, \\ 1 & \text{if } (\alpha_{i,d}|\alpha_{i,d}) = 0. \end{cases}$$

Then we extend p to the $\mathbb{Z}$-linear function $p\colon Q_d \to \mathbb{Z}$.

3 The quantum affine superalgebras of type $D^{(1)}(2, 1; x)$

In this section, we define the quantum affine superalgebra of type $D^{(1)}(2, 1; x)$, where $x \in \mathbb{C} \setminus \{0, -1\}$. Fix an element $\hbar \in \mathbb{C} \setminus \mathbb{Z}\pi\sqrt{-1}$ such that $\exp(\hbar k a) \neq 1$ for all $k \in \mathbb{N}$ and all $a \in \{1, x, x + 1\}$. We set $q^u := \exp(\hbar u)$ and $[u]_q := (q^u - q^{-u})/(q - q^{-1})$ for any $u \in \mathbb{C}$, where $q := q^1$. Note that $q^{ka} \neq 1$ for all $k \in \mathbb{N}$ and all $a \in \{1, x, x + 1\}$.

First, for each $d \in \mathcal{D}$, we define the associative $\mathbb{C}$-algebra $\mathcal{U}'_d$ with the unit 1 by the generators

$$\sigma_d,\ K_{i,d}^{\pm\frac{1}{2}},\ E_{i,d},\ F_{i,d} \quad (i \in \mathrm{I}),$$

and the following relations

$$XY = YX \quad \text{for} \quad X, Y \in \{\sigma_d, K_{i,d}^{\pm\frac{1}{2}}\}, \tag{3.1}$$

$$\sigma_d^2 = 1, \quad K_{i,d}^{\frac{1}{2}} K_{i,d}^{-\frac{1}{2}} = K_{i,d}^{-\frac{1}{2}} K_{i,d}^{\frac{1}{2}} = 1, \tag{3.2}$$

$$\sigma_d E_{i,d} \sigma_d = (-1)^{p(\alpha_{i,d})} E_{i,d}, \quad \sigma_d F_{i,d} \sigma_d = (-1)^{p(\alpha_{i,d})} F_{i,d}, \tag{3.3}$$

$$K_{i,d}^{\frac{1}{2}} E_{j,d} K_{i,d}^{-\frac{1}{2}} = q^{(\alpha_{i,d}|\alpha_{j,d})/2} E_{j,d}, \quad K_{i,d}^{\frac{1}{2}} F_{j,d} K_{i,d}^{-\frac{1}{2}} = q^{-(\alpha_{i,d}|\alpha_{j,d})/2} F_{j,d}, \tag{3.4}$$

$$E_{i,d} F_{j,d} - (-1)^{p(\alpha_{i,d})p(\alpha_{j,d})} F_{j,d} E_{i,d} = \delta_{ij} \{(K_{i,d}^{\frac{1}{2}})^2 - (K_{i,d}^{-\frac{1}{2}})^2\}/(q - q^{-1}), \tag{3.5}$$

for all $i, j \in \mathrm{I}$. In the following we use notations

$$E_{\alpha_{i,d}} := E_{i,d}, \qquad F_{\alpha_{i,d}} := F_{i,d}, \qquad K_\lambda := \Pi_{i \in \mathrm{I}} K_{i,d}^{\frac{1}{2} m_i},$$

where $\lambda = \frac{1}{2} \sum_{i \in \mathrm{I}} m_i \alpha_{i,d} \in \frac{1}{2} Q_d$ with $m_i \in \mathbb{Z}$. As we will see later, the quantum affine superalgebra is obtained from the quotient of $\mathcal{U}'_d$ divided by the Serre like relations. Still it will be convenient to work with the algebra $\mathcal{U}'_d$. The algebra $\mathcal{U}'_d$ has a unique Q_d-grading

$$\mathcal{U}'_d = \bigoplus_{\lambda \in Q_d} \mathcal{U}'_{d,\lambda}, \qquad \mathcal{U}'_{d,\lambda} \mathcal{U}'_{d,\mu} \subset \mathcal{U}'_{d,\lambda+\mu}$$

such that $\{1, \sigma_d, K_{i,d}^{\pm\frac{1}{2}}\} \subset \mathcal{U}'_{d,0}$, $E_{i,d} \in \mathcal{U}'_{d,\alpha_{i,d}}$, and $F_{i,d} \in \mathcal{U}'_{d,-\alpha_{i,d}}$ for all $i \in \mathrm{I}$. Note that there exists a unique algebra automorphism Ψ_d of $\mathcal{U}'_d$ such that

$$\Psi_d(\sigma_d) = \sigma_d, \quad \Psi_d(K_{i,d}^{\pm\frac{1}{2}}) = K_{i,d}^{\mp\frac{1}{2}}, \quad \Psi_d(E_{i,d}) = (-1)^{p(\alpha_{i,d})} F_{i,d}, \quad \Psi_d(F_{i,d}) = E_{i,d}.$$

To state the Serre relation, we need to introduce the q-super-bracket $[\![\,,\,]\!]$ for the elements of $\mathcal{U}'_d$. For $a \in \mathbb{C}$, $X_\lambda \in \mathcal{U}'_{d,\lambda}$, and $X_\mu \in \mathcal{U}'_{d,\mu}$ with $\lambda, \mu \in Q_d$, we set

$$[X_\lambda, X_\mu]_a := X_\lambda X_\mu - (-1)^{p(\lambda)p(\mu)} a\, X_\mu X_\lambda, \tag{3.6}$$

$$[\![X_\lambda, X_\mu]\!] := [X_\lambda, X_\mu]_{q^{-(\lambda|\mu)}}. \tag{3.7}$$

Then we extend the $[\,,\,]_a$ and $[\![\,,\,]\!]$ to the bilinear mappings $\mathcal{U}'_d \times \mathcal{U}'_d \to \mathcal{U}'_d$ respectively.

Now we define the quantum affine superalgebras U'_d of $D^{(1)}(2, 1; x)$ for Dynkin diagrams labeled by $d \in \mathcal{D}$.

Definition 3.1. The quantum affine superalgebra U'_d of type $D^{(1)}(2,1;x)$ over $\mathbb{C}$ is the quotient algebra of $\mathcal{U}'_d$ divided by the two-sided ideal generated by the following elements:

$$E_{i,d}^2, \quad \text{where } i \in \mathrm{I} \text{ and } p(\alpha_{i,d}) = 1, \tag{3.8}$$
$$[\![E_{i,d}, E_{j,d}]\!], \quad \text{where } i,j \in \mathrm{I},\ i \neq j, \text{ and } (\alpha_{i,d}|\alpha_{j,d}) = 0, \tag{3.9}$$
$$[\![E_{i,d}, [\![E_{i,d}, E_{j,d}]\!]]\!], \quad \text{where } i,j \in \mathrm{I},\ i \neq j, \text{ and } p(\alpha_{i,d}) = 0, \text{ and } (\alpha_{i,d}|\alpha_{j,d}) \neq 0, \tag{3.10}$$
$$[(\alpha_{i,4}|\alpha_{k,4})]_q [\![[\![E_{i,4}, E_{j,4}]\!], E_{k,4}]\!] - [(\alpha_{i,4}|\alpha_{j,4})]_q [\![[\![E_{i,4}, E_{k,4}]\!], E_{j,4}]\!], \quad \text{if } d = 4, \text{ where } i,j,k \in \mathrm{I} \text{ such that } i < j < k, \tag{3.11}$$
$$\begin{aligned}&[(\alpha_{i,d}+\alpha_{d,d}|\alpha_{k,d}+\alpha_{d,d})]_q [\![[\![[\![E_{d,d}, E_{i,d}]\!], [\![E_{d,d}, E_{j,d}]\!]]\!], [\![E_{d,d}, E_{k,d}]\!]]\!] \\ &\quad - [(\alpha_{i,d}+\alpha_{d,d}|\alpha_{j,d}+\alpha_{d,d})]_q [\![[\![[\![E_{d,d}, E_{i,d}]\!], [\![E_{d,d}, E_{k,d}]\!]]\!], [\![E_{d,d}, E_{j,d}]\!]]\!] \\ &\qquad \text{if } d \neq 4, \text{ where } \{i,j,k,d\} = \mathrm{I}, \text{ and } i < j < k,\end{aligned} \tag{3.12}$$
$$\Psi_d(X), \quad \text{for all } X \text{ in the above.} \tag{3.13}$$

Because of history, we call the above relations the Serre relations.

Since each element displayed in (3.8)–(3.13) is an element of $\mathcal{U}'_{d,\lambda}$ for some $\lambda \in Q_d$, the Q_d-grading of $\mathcal{U}'_d$ induces the Q_d-grading $U'_d = \oplus_{\lambda \in Q_d} U'_{d,\lambda}$. We call a non-zero element $x \in \mathcal{U}'_d$ (resp. $x \in U'_d$) a *weight vector* with weight λ if $x \in \mathcal{U}'_{d,\lambda}$ (resp. $x \in U'_{d,\lambda}$), and set $\mathrm{wt}(x) = \lambda$. To simplify notations, we will also write $\mathrm{wt}\, x$ instead of $\mathrm{wt}(x)$. The linear mappings $[\,,\,]_a$ and $[\![\,,\,]\!]$ from $U'_d \times U'_d$ to U'_d can be defined by the same way as above, and the Ψ_d induces an automorphism of U'_d, which will also denoted by Ψ_d.

Let $\mathcal{U}'^{>0}_d$, $\mathcal{U}'^{0}_d$, and $\mathcal{U}'^{<0}_d$ be the subalgebras of $\mathcal{U}'_d$ generated by the sets $\{E_{i,d} \,|\, i \in \mathrm{I}\}$, $\{\sigma_d, K_{i,d}^{\pm\frac{1}{2}} \,|\, i \in \mathrm{I}\}$, and $\{F_{i,d} \,|\, i \in \mathrm{I}\}$, respectively, and set $\mathcal{U}'^{\geq 0}_d := \mathcal{U}'^{>0}_d \mathcal{U}'^{0}_d$ and $\mathcal{U}'^{\leq 0}_d := \mathcal{U}'^{0}_d \mathcal{U}'^{<0}_d$. Let $U'^{>0}_d$, U'^{0}_d, $U'^{<0}_d$, $U'^{\geq 0}_d$, and $U'^{\leq 0}_d$ be the images of $\mathcal{U}'^{>0}_d$, $\mathcal{U}'^{0}_d$, $\mathcal{U}'^{<0}_d$, $\mathcal{U}'^{\geq 0}_d$, and $\mathcal{U}'^{\leq 0}_d$, respectively, under the canonical projection $\mathcal{U}'_d \to U'_d$.

Theorem 3.2 ([4]). (1) *The associative $\mathbb{C}$-algebras $\mathcal{U}'_d$ and U'_d can be regarded as Hopf algebras $(\mathcal{U}'_d, \Delta, \varepsilon, S)$ and $(U'_d, \Delta, \varepsilon, S)$ such that*

$$\Delta(X) = X \otimes X, \quad \Delta(E_{i,d}) = E_{i,d} \otimes 1 + K_{i,d}\sigma_d^{p(\alpha_{i,d})} \otimes E_{i,d}, \quad \Delta(F_{i,d}) = F_{i,d} \otimes K_{i,d}^{-1} + \sigma_d^{p(\alpha_{i,d})} \otimes F_{i,d}, \tag{3.14}$$
$$\varepsilon(X) = 1, \quad \varepsilon(E_{i,d}) = 0, \quad \varepsilon(F_{i,d}) = 0, \tag{3.15}$$
$$S(X) = X^{-1}, \quad S(E_{i,d}) = -K_{i,d}^{-1}\sigma_d^{p(\alpha_{i,d})} E_{i,d}, \quad S(F_{i,d}) = -(-1)^{p(\alpha_{i,d})} F_{i,d} K_{i,d} \sigma_d^{p(\alpha_{i,d})}, \tag{3.16}$$

where $i \in \mathrm{I}$ and $X \in \{\sigma_d, K_{i,d}^{\pm\frac{1}{2}} \,|\, i \in \mathrm{I}\}$.

(2) *The multiplication $X \otimes Y \otimes Z \mapsto XYZ$ defines the following isomorphisms of Q_d-graded $\mathbb{C}$-vector spaces:*

$$\mathcal{U}'^{>0}_d \otimes \mathcal{U}'^{0}_d \otimes \mathcal{U}'^{<0}_d \simeq \mathcal{U}'_d, \qquad U'^{>0}_d \otimes U'^{0}_d \otimes U'^{<0}_d \simeq U'_d.$$

Moreover, the algebra $\mathcal{U}'^{>0}_d$ (resp. $\mathcal{U}'^{<0}_d$) is the free algebra generated by the set $\{E_{i,d} \,|\, i \in \mathrm{I}\}$ (resp. $\{F_{i,d} \,|\, i \in \mathrm{I}\}$), and the algebra $U'^{>0}_d$ (resp. $U'^{<0}_d$) is isomorphic to the quotient algebra of $\mathcal{U}'^{>0}_d$ (resp. $\mathcal{U}'^{<0}_d$) divided by the two-sided ideal generated by the elements displayed in (3.8)–(3.12) (resp. (3.13)). The both $\mathcal{U}'^{0}_d$ and U'^{0}_d are isomorphic to the commutative algebra defined by the generators $\{\sigma_d, K_{i,d}^{\pm\frac{1}{2}} \,|\, i \in \mathrm{I}\}$ and the relations (3.2).

Proposition 3.3 ([8]). (1) *Let (Δ, ε, S) be the Hopf algebra structure on $\mathcal{U}'_d$ introduced in Theorem 3.2. Let $\mathcal{SR} \in \mathcal{U}'^{>0}_d$ be an arbitrary element displayed in (3.8)–(3.11), and set $\mathcal{SR}^- := \Psi_d(\mathcal{SR})$. Then the following equalities hold:*

$$\Delta(\mathcal{SR}) = \mathcal{SR} \otimes 1 + \sigma_d^{p(\mathrm{wt}(\mathcal{SR}))} K_{\mathrm{wt}(\mathcal{SR})} \otimes \mathcal{SR}, \qquad \Delta(\mathcal{SR}^-) = \mathcal{SR}^- \otimes K_{\mathrm{wt}(\mathcal{SR})}^{-1} + \sigma_d^{p(\mathrm{wt}(\mathcal{SR}))} \otimes \mathcal{SR}^-, \tag{3.17}$$
$$S(\mathcal{SR}) = -\sigma_d^{\mathrm{wt}(\mathcal{SR})} K_{\mathrm{wt}(\mathcal{SR})}^{-1} \mathcal{SR}, \qquad S(\mathcal{SR}^-) = -(-1)^{\mathrm{wt}(\mathcal{SR})} \mathcal{SR}^- \sigma_d^{\mathrm{wt}(\mathcal{SR})} K_{\mathrm{wt}(\mathcal{SR})}. \tag{3.18}$$

(2) *Let $\mathcal{L}$ be the two-sided ideal of $\mathcal{U}'^{\geq 0}_d$ generated by the elements displayed in (3.8) and (3.9). If $\mathcal{SR} \in \mathcal{U}'^{>0}_d$ is an arbitrary element displayed in (3.12), then the left (resp. right) equality of (3.17) holds modulo $\mathcal{L} \otimes \mathcal{U}'^{\geq 0}_d + \mathcal{U}'^{\geq 0}_d \otimes \mathcal{L}$ (resp. $\Psi_d(\mathcal{L}) \otimes \mathcal{U}'^{\leq 0}_d + \mathcal{U}'^{\leq 0}_d \otimes \Psi_d(\mathcal{L})$), and the left (resp. right) equality of (3.18) holds modulo $\mathcal{L}$ (resp. $\Psi_d(\mathcal{L})$).*

4 The bilinear forms

In this section, we construct the bilinear form on the quantum affine superalgebra of type $D^{(1)}(2,1;x)$.

Let $I=(\beta_1,\cdots,\beta_n)$ be a finite sequences of elements of Π_d $(d\in\mathcal{D})$ with $n\in\mathbb{N}$. Then we set $|I|:=n$ and call $|I|$ the *length* of I. We define $E_I\in\mathcal{U}_d'^{>0}$, $E_I\in U_d'^{>0}$, $F_I\in\mathcal{U}_d'^{<0}$, and $F_I\in U_d'^{<0}$ by setting

$$E_I:=E_{\beta_1}\cdots E_{\beta_n},\qquad F_I:=F_{\beta_1}\cdots F_{\beta_n},\tag{4.1}$$

and set $\mathrm{wt}(I):=\mathrm{wt}(E_I)=\sum_{i=1}^n\beta_i$. In the case where $I=\emptyset$, we set $E_\emptyset=F_\emptyset:=1$ and $\mathrm{wt}(\emptyset):=0$. To simplify notations, we will also write $\mathrm{wt}I$ instead of $\mathrm{wt}(I)$. Let $J=(\gamma_1,\cdots,\gamma_m)$ be a finite sequences of elements of Π_d with $n\geq m$. If there exists a subset $\{i_1,\cdots,i_m\}$ of $\{1,\cdots,n\}$ such that $i_1<\cdots<i_m$ and $J=(\beta_{i_1},\cdots,\beta_{i_m})$, then we call J a *subsequence* of I. Moreover, for a subsequence K of I, there exists a subset $\{j_1,\cdots,j_{n-m}\}$ of $\{1,\cdots,n\}$ such that $\{i_1,\cdots,i_m\}\amalg\{j_1,\cdots,j_{n-m}\}=\{1,\cdots,n\}$ with $j_1<\cdots<j_{n-m}$ and $K=(\beta_{j_1},\cdots,\beta_{j_{n-m}})$, then we write $I=J+K$. Especially, if $i_m<j_1$, we denote $I=(J,K)$.

Lemma 4.1. *Let I be a finite sequence of elements of Π_d. Then there exist elements $c^I_{A,B}(q)\in\mathbb{Z}[q^{\pm1},q^{\pm x}]$ such that in $\mathcal{U}_d'$ and U_d':*

$$\Delta(E_I)=\sum_{A,B}c^I_{A,B}(q)E_A\sigma_d^{p(\mathrm{wt}B)}K_B\otimes E_B,\tag{4.2}$$

$$\Delta(F_I)=\sum_{A,B}c^I_{A,B}(q^{-1})F_A\sigma_d^{p(\mathrm{wt}B)}\otimes K^{-1}_{\mathrm{wt}A}F_B=\sum_{A}c^I_{A,B}(q^{-1})q^{(\mathrm{wt}A|\mathrm{wt}B)}F_A\sigma_d^{p(\mathrm{wt}B)}\otimes F_BK^{-1}_{\mathrm{wt}A},\tag{4.3}$$

where the sum is over all subsequences A, B of Π_d with $I=A+B$. Moreover, one has $c^I_{I,\emptyset}=c^I_{\emptyset,I}=1$.

Proof. We use the induction on the length of I. In the case where $I=\emptyset$, we have $E_I=F_I=1$ and $\Delta(E_I)=\Delta(F_I)=1\otimes1$, so the claim holds with $c^\emptyset_{\emptyset,\emptyset}=1$. Suppose that the formulas hold for some I, let $\alpha\in\Pi_d$ and consider $I':=(\alpha,I)$. Then we have

$$\begin{aligned}\Delta(E_{I'})&=(E_\alpha\otimes1+\sigma_d^{p(\alpha)}K_\alpha\otimes E_\alpha)(\sum_{A,B}c^I_{A,B}(q)E_A\sigma_d^{p(\mathrm{wt}B)}K_{\mathrm{wt}B}\otimes E_B)\\&=\sum_{A,B}(c^I_{A,B}(q)E_{(\alpha,A)}\sigma_d^{p(\mathrm{wt}B)}K_{\mathrm{wt}B}\otimes E_B+c^I_{A,B}(q)\sigma_d^{p(\alpha)}K_\alpha E_A\sigma_d^{p(\mathrm{wt}B)}K_{\mathrm{wt}B}\otimes E_{(\alpha,B)})\\&=\sum_{A,B}(c^I_{A,B}(q)E_{(\alpha,A)}\sigma_d^{p(\mathrm{wt}B)}K_{\mathrm{wt}B}\otimes E_B+(-1)^{p(\alpha)p(\mathrm{wt}A)}c^I_{A,B}(q)q^{(\alpha|\mathrm{wt}A)}E_A\sigma_d^{p(\alpha+\mathrm{wt}B)}K_{\alpha+\mathrm{wt}B}\otimes E_{(\alpha,B)})\end{aligned}$$

and

$$\begin{aligned}\Delta(F_{I'})&=(F_\alpha\otimes K_\alpha^{-1}+\sigma_d^{p(\alpha)}\otimes F_\alpha)(\sum_{A}c^I_{A,B}(q^{-1})F_A\sigma_d^{p(\mathrm{wt}B)}\otimes K^{-1}_{\mathrm{wt}A}F_B)\\&=\sum_{A}(c^I_{A,B}(q^{-1})F_{(\alpha,A)}\sigma_d^{p(\mathrm{wt}B)}\otimes K^{-1}_{\alpha+\mathrm{wt}A}F_B+c^I_{A,B}(q^{-1})\sigma_d^{p(\alpha)}F_A\sigma_d^{p(\mathrm{wt}B)}\otimes F_\alpha K^{-1}_{\mathrm{wt}A}F_{\mathrm{wt}B})\\&=\sum_{A}(c^I_{A,B}(q^{-1})F_{(\alpha,A)}\sigma_d^{p(\mathrm{wt}B)}\otimes K^{-1}_{\alpha+\mathrm{wt}A}F_B+(-1)^{p(\alpha)p(\mathrm{wt}A)}c^I_{A,B}(q^{-1})q^{-(\alpha|\mathrm{wt}A)}F_A\sigma_d^{p(\alpha+\mathrm{wt}B)}\otimes K^{-1}_{\mathrm{wt}A}F_{(\alpha,B)}).\end{aligned}$$

Let us define $c^{I'}_{A',B'}(q)$ as follows. If both A' and B' begin with α, then write $A'=(\alpha,A)$ and $B'=(\alpha,B)$ and set $c^{I'}_{A',B'}(q):=c^I_{A,B'}(q)+(-1)^{p(\alpha)p(\mathrm{wt}A')}c^I_{A',B}(q)q^{(\alpha|\mathrm{wt}A')}$. If A' begins with α and B' does not, then set $c^{I'}_{A',B'}(q):=c^I_{A,B'}(q)$, where $A'=(\alpha,A)$. If B' begins with α and A does not, then set $c^{I'}_{A',B'}(q):=(-1)^{p(\alpha)p(\mathrm{wt}A')}c^I_{A',B}(q)q^{(\alpha|\mathrm{wt}A')}$, where $B'=(\alpha,B)$. If both A' and B' do not begin with α, then set $c^{I'}_{A',B'}(q):=0$. Then it is easy to check the $c^{I'}_{A',B'}(q)$'s satisfy the claims. □

Let t be an indeterminate, and define $[n]_t$, $[n]_t!$, and $\begin{bmatrix}n\\i\end{bmatrix}_t$ to be elements of $\mathbb{Z}[t,t^{-1}]$ by setting

$$[n]_t:=\frac{t^n-t^{-n}}{t-t^{-1}},\qquad [n]_t!:=[n]_t[n-1]_t\cdots[1]_t,\qquad \begin{bmatrix}n\\m\end{bmatrix}_t:=\frac{[n]_t!}{[n-m]_t![m]_t!}$$

for each $n,m\in\mathbb{Z}_{\geq0}$ with $n\geq m$, where $[0]_t!:=1$. For each $\alpha\in\Pi_d$, we set

$$q_\alpha:=(\sqrt{-1})^{p(\alpha)}q^{(\alpha|\alpha)/2},\qquad \overline{q_\alpha}:=(\sqrt{-1})^{p(\alpha)}q^{-(\alpha|\alpha)/2}=(-1)^{p(\alpha)}q_\alpha^{-1}.\tag{4.4}$$

By using the substitutions $t = q_\alpha$ and $t = \overline{q_\alpha}$, we define $[n]_{q_\alpha}$, $[n]_{\overline{q_\alpha}}$, $[n]_{q_\alpha}!$, $[n]_{\overline{q_\alpha}}!$, $\begin{bmatrix} n \\ m \end{bmatrix}_{q_\alpha}$, and $\begin{bmatrix} n \\ m \end{bmatrix}_{\overline{q_\alpha}}$. Then we note that

$$[n]_{\overline{q_\alpha}} = (-1)^{(n-1)p(\alpha)}[n]_{q_\alpha}, \qquad \begin{bmatrix} n \\ m \end{bmatrix}_{\overline{q_\alpha}} = (-1)^{m(n-1)p(\alpha)}\begin{bmatrix} n \\ m \end{bmatrix}_{q_\alpha} = (-1)^{m(n-m)p(\alpha)}\begin{bmatrix} n \\ m \end{bmatrix}_{q_\alpha} \tag{4.5}$$

for all $n, m \in \mathbb{Z}_{\geq 0}$ with $n \geq m$. Moreover, note that $q_\alpha^2 = (-1)^{p(\alpha)}q^{(\alpha|\alpha)}$, hence

$$\sigma_d K_\alpha E_\alpha = q_\alpha^2 E_\alpha \sigma_d K_\alpha, \qquad \sigma_d K_\alpha F_\alpha = q_\alpha^{-2} F_\alpha \sigma_d K_\alpha. \tag{4.6}$$

Lemma 4.2. *Let $\alpha \in \Pi_d$, and $n \in \mathbb{N}$. Then the following equalities hold in $\mathcal{U}'_d$ and U'_d:*

$$\Delta(E_\alpha^n) = \sum_{i=0}^{n} q_\alpha^{i(n-i)}\begin{bmatrix} n \\ i \end{bmatrix}_{q_\alpha} E_\alpha^{n-i}\sigma_d^{ip(\alpha)}K_\alpha^i \otimes E_\alpha^i, \tag{4.7}$$

$$\Delta(F_\alpha^n) = \sum_{i=0}^{n} (-1)^{i(n-i)p(\alpha)}q_\alpha^{i(n-i)}\begin{bmatrix} n \\ i \end{bmatrix}_{q_\alpha} F_\alpha^i\sigma_d^{(n-i)p(\alpha)} \otimes F_\alpha^{n-i}K_\alpha^{-i}. \tag{4.8}$$

Proof. We use the induction on n. In the case where $n = 1$, the claims are clear. Suppose that the formulas hold for some n. Then, by the center equality in (3.14), we see that

$$\begin{aligned}
\Delta(E_\alpha^{n+1}) &= (E_\alpha \otimes 1 + \sigma_d^{p(\alpha)}K_\alpha \otimes E_\alpha)\Big(\sum_{i=0}^{n} q_\alpha^{i(n-i)}\begin{bmatrix} n \\ i \end{bmatrix}_{q_\alpha} E_\alpha^{n-i}\sigma_d^{ip(\alpha)}K_\alpha^i \otimes E_\alpha^i\Big) \\
&= \sum_{i=0}^{n} q_\alpha^{i(n-i)}\begin{bmatrix} n \\ i \end{bmatrix}_{q_\alpha} E_\alpha^{n+1-i}\sigma_d^{ip(\alpha)}K_\alpha^i \otimes E_\alpha^i + \sum_{i=0}^{n} q_\alpha^{i(n-i)}\begin{bmatrix} n \\ i \end{bmatrix}_{q_\alpha} K_\alpha\sigma_d^{p(\alpha)}E_\alpha^{n-i}K_\alpha^i\sigma_d^{ip(\alpha)} \otimes E_\alpha^{i+1} \\
&= \sum_{i=0}^{n} q_\alpha^{i(n-i)}\begin{bmatrix} n \\ i \end{bmatrix}_{q_\alpha} E_\alpha^{n+1-i}\sigma_d^{ip(\alpha)}K_\alpha^i \otimes E_\alpha^i + \sum_{i=0}^{n} q_\alpha^{i(n-i)}\begin{bmatrix} n \\ i \end{bmatrix}_{q_\alpha} q_\alpha^{2(n-i)}E_\alpha^{n-i}\sigma_d^{(i+1)p(\alpha)}K_\alpha^{i+1} \otimes E_\alpha^{i+1} \\
&= \sum_{i=0}^{n} q_\alpha^{i(n-i)}\begin{bmatrix} n \\ i \end{bmatrix}_{q_\alpha} E_\alpha^{n+1-i}\sigma_d^{ip(\alpha)}K_\alpha^i \otimes E_\alpha^i + \sum_{i=1}^{n+1} q_\alpha^{(i+1)(n+1-i)}\begin{bmatrix} n \\ i-1 \end{bmatrix}_{q_\alpha} E_\alpha^{n+1-i}\sigma_d^{ip(\alpha)}K_\alpha^i \otimes E_\alpha^i \\
&= \sum_{i=0}^{n+1} q_\alpha^{i(n+1-i)}\Big(q_\alpha^{-i}\begin{bmatrix} n \\ i \end{bmatrix}_{q_\alpha} + q_\alpha^{n-i+1}\begin{bmatrix} n \\ i-1 \end{bmatrix}_{q_\alpha}\Big)E_\alpha^{n+1-i}\sigma_d^{ip(\alpha)}K_\alpha^i \otimes E_\alpha^i \\
&= \sum_{i=0}^{n+1} q_\alpha^{i(n+1-i)}\begin{bmatrix} n+1 \\ i \end{bmatrix}_{q_\alpha} E_\alpha^{n+1-i}\sigma_d^{ip(\alpha)}K_\alpha^i \otimes E_\alpha^i.
\end{aligned}$$

By (4.7) and the second equality in (4.3), we see that

$$\begin{aligned}
\Delta(F_\alpha^n) &= \sum_{i=0}^{n} (\overline{q_\alpha})^{i(n-i)}\begin{bmatrix} n \\ i \end{bmatrix}_{\overline{q_\alpha}} q^{i(n-i)(\alpha|\alpha)}F_\alpha^i\sigma_d^{(n-i)p(\alpha)} \otimes F_\alpha^{n-i}K_\alpha^{-i} \\
&= \sum_{i=0}^{n} (-1)^{i(n-i)p(\alpha)}q_\alpha^{-i(n-i)}(-1)^{i(n-i)p(\alpha)}\begin{bmatrix} n \\ i \end{bmatrix}_{q_\alpha} (-1)^{i(n-i)p(\alpha)}q_\alpha^{2i(n-i)}F_\alpha^i\sigma_d^{(n-i)p(\alpha)} \otimes F_\alpha^{n-i}K_\alpha^{-i} \\
&= \sum_{i=0}^{n} (-1)^{i(n-i)p(\alpha)}q_\alpha^{i(n-i)}\begin{bmatrix} n \\ i \end{bmatrix}_{q_\alpha} F_\alpha^i\sigma_d^{(n-i)p(\alpha)} \otimes F_\alpha^{n-i}K_\alpha^{-i}. \quad \square
\end{aligned}$$

We recall the following fact. Let A be a bialgebra over a field $\mathbb{K}$ with Δ the coproduct and ε the counit, i.e., A is an associative algebra over $\mathbb{K}$ with algebra homomorphisms $\Delta\colon A \to A \otimes_{\mathbb{K}} A$ and $\varepsilon\colon A \to \mathbb{K}$ such that

$$(\Delta \otimes id_A) \circ \Delta = (id_A \otimes \Delta) \circ \Delta, \qquad (\varepsilon \otimes id_A) \circ \Delta = (id_A \otimes \varepsilon) \circ \Delta = id_A.$$

Then the dual space $A^* = \mathrm{Hom}_{\mathbb{K}}(A, \mathbb{K})$ is naturally regarded as an associative algebra with the unit over $\mathbb{K}$ as follows. Let $f, g \in A^*$. Then the product $fg \in A^*$ is defined by

$$fg(a) = (f \otimes g)(\Delta(a)),$$

where $a \in A$. The counit ε is the unit of A^*.

Definition 4.3. For each $\alpha \in \Pi_d$, we define a linear form f_α on $U_d'^{\geq 0}$ by setting

$$f_\alpha(E_I \sigma_d^m K_\lambda) := \begin{cases} 1/(q_\alpha^{-1} - q_\alpha) & \text{if } I = (\alpha), \\ 0 & \text{otherwise.} \end{cases} \tag{4.9}$$

For each sequence $I = (\beta_1, \beta_2, \dots, \beta_p)$ of elements of Π_d, we set

$$f_I := f_{\beta_1} f_{\beta_2} \cdots f_{\beta_p}. \tag{4.10}$$

If $I = \emptyset$, we set

$$f_\emptyset(E_J \sigma_d^n K_\mu) := \delta_{J,\emptyset}. \tag{4.11}$$

For each $m \in \{0, 1\}$ and $\lambda \in \frac{1}{2}Q_d$, we define a linear form $k_{m,\lambda}$ on $U_d'^{\geq 0}$ by setting

$$k_{m,\lambda}(E_I \sigma_d^n K_\mu) := \begin{cases} (-1)^{mn} q^{-(\lambda|\mu)} & \text{if } I = \emptyset, \\ 0 & \text{otherwise.} \end{cases} \tag{4.12}$$

Lemma 4.4. *Let I, J be arbitrary sequences of elements of Π_d, $m, n \in \{0, 1\}$, and $\lambda, \mu \in \frac{1}{2}Q_d$. Then*

$$f_I k_{m,\lambda}(E_J \sigma_d^n K_\mu) = (-1)^{mn} q^{-(\lambda|\mu)} f_I(E_J). \tag{4.13}$$

Moreover, if $\mathrm{wt}(I) \neq \mathrm{wt}(J)$, then $f_I(E_J) = 0$.

Proof. Firstly, we consider the case where $m = 0$ and $\lambda = 0$. Then it is clear that $f_I k_{m,\lambda} = f_I$. We use the induction on the length $|I|$. In the case where $|I| = 0$, i.e., $I = \emptyset$, both claims follow from (4.11). In the case where $|I| = 1$, i.e., $I = (\alpha)$ for some $\alpha \in \Pi_d$, both claims follow from (4.9). Suppose that the claims hold for some I and consider $I' = (\alpha, I)$. By the definition of the product on $(\mathcal{U}'^{\leq 0})^{\otimes 2}$ and Lemma 4.1, we see that

$$\begin{aligned} f_{I'}(E_J \sigma_d^n K_\mu) &= f_\alpha \otimes f_I(\Delta(E_J \sigma_d^n K_\mu)) = f_\alpha \otimes f_I\Big(\sum_{A,B} c_{A,B}^J(q) E_A \sigma_d^{p(\mathrm{wt}B)+n} K_{\mathrm{wt}(B)+\mu} \otimes E_B \sigma_d^n K_\mu\Big) \\ &= \sum_{A,B} c_{A,B}^J(q) f_\alpha(E_A \sigma_d^{p(\mathrm{wt}B)+n} K_{\mathrm{wt}(B)+\mu}) f_I(E_B \sigma_d^n K_\mu) = \sum_{A,B} c_{A,B}^J(q) f_\alpha(E_A) f_I(E_B). \end{aligned}$$

By the previous equality, we see that $f_{I'}(E_J \sigma_d^n K_\mu) = f_{I'}(E_J)$. Here we suppose that $f_{I'}(E_J) \neq 0$. Then there exist A, B such that $c_{A,B}^J(q) \neq 0$ and $f_\alpha(E_A) f_I(E_B) \neq 0$. The condition $f_\alpha(E_A) f_I(E_B) \neq 0$ implies that $A = (\alpha)$ and $\mathrm{wt}(B) = \mathrm{wt}(I)$. Thus we get that $\mathrm{wt}(J) = \mathrm{wt}(A) + \mathrm{wt}(B) = \alpha + \mathrm{wt}(I) = \mathrm{wt}(I')$.

Secondly, we prove (4.13) in the general case. By the previous result and (4.12), we see that

$$\begin{aligned} f_I k_{m,\lambda}(E_J \sigma_d^n K_\mu) &= f_I \otimes k_{m,\lambda}(\Delta(E_J \sigma_d^n K_\mu)) = f_I \otimes k_{m,\lambda}\Big(\sum_{A,B} c_{A,B}^J(q) E_A \sigma_d^{p(\mathrm{wt}B)+n} K_{\mathrm{wt}(B)+\mu} \otimes E_B \sigma_d^n K_\mu\Big) \\ &= \sum_{A,B} c_{A,B}^J(q) f_I(E_A \sigma_d^{p(\mathrm{wt}B)+n} K_{\mathrm{wt}(B)+\mu}) k_{m,\lambda}(E_B \sigma_d^n K_\mu) \\ &= f_I(E_J \sigma_d^{p(\mathrm{wt}J)+n} K_{\mathrm{wt}(J)+\mu}) \cdot (-1)^{mn} q^{-(\lambda|\mu)} = (-1)^{mn} q^{-(\lambda|\mu)} f_I(E_J). \quad \square \end{aligned}$$

Lemma 4.5. *Let I be an arbitrary sequence of elements of Π_d, $m, n \in \{0, 1\}$, and $\lambda, \mu \in \frac{1}{2}Q_d$. Then*

$$k_{m,\lambda} k_{n,\mu} = k_{m+n,\lambda+\mu}, \tag{4.14}$$

$$k_{m,\lambda} f_I = (-1)^{mp(\mathrm{wt}I)} q^{-(\lambda|\mathrm{wt}(I))} f_I k_{m,\lambda}. \tag{4.15}$$

Proof. We see that

$$\begin{aligned} k_{m,\lambda} k_{n,\mu}(E_I \sigma_d^l K_\nu) &= k_{m,\lambda} \otimes k_{n,\mu}\Big(\sum_{A,B} c_{A,B}^I(q) E_A \sigma_d^{p(\mathrm{wt}B)+l} K_{\mathrm{wt}(B)+\nu} \otimes E_B \sigma_d^l K_\nu\Big) \\ &= \sum_{A,B} c_{A,B}^I(q) k_{m,\lambda}(E_A \sigma_d^{p(\mathrm{wt}B)+l} K_{\mathrm{wt}(B)+\nu}) k_{n,\mu}(E_B \sigma_d^l K_\nu) \\ &= \begin{cases} (-1)^{ml} q^{-(\lambda|\nu)} \cdot (-1)^{nl} q^{-(\mu|\nu)} & \text{if } I = \emptyset, \\ 0 & \text{otherwise} \end{cases} = k_{m+n,\lambda+\mu}(E_I \sigma_d^l K_\nu). \end{aligned}$$

Thus we get (4.14). By Lemma 4.4, we see that

$$\begin{aligned}
k_{m,\lambda} f_I(E_J \sigma_d^n K_\mu) &= k_{m,\lambda} \otimes f_I(\Delta(E_J \sigma_d^n K_\mu)) \\
&= k_{m,\lambda} \otimes f_I\Big(\sum_{A,B} c^J_{A,B}(q) E_A \sigma_d^{p(\mathrm{wt} B)+n} K_{\mathrm{wt}(B)+\mu} \otimes E_B \sigma_d^n K_\mu\Big) \\
&= \sum_{A,B} c^J_{A,B}(q) k_{m,\lambda}(E_A \sigma_d^{p(\mathrm{wt} B)+n} K_{\mathrm{wt}(B)+\mu}) f_I(E_B \sigma_d^n K_\mu) \\
&= k_{m,\lambda}(\sigma_d^{p(\mathrm{wt} J)+n} K_{\mathrm{wt}(J)+\mu}) f_I(E_J) = (-1)^{mp(\mathrm{wt} J)+n)} q^{-(\lambda | \mathrm{wt}(J)+\mu)} f_I(E_J). \\
&= \delta_{\mathrm{wt}(I),\mathrm{wt}(J)} (-1)^{mp(\mathrm{wt} I)} q^{-(\lambda|\mathrm{wt} I)} f_I k_{m,\lambda}(E_J \sigma_d^n K_\mu).
\end{aligned}$$

Thus we get (4.15). □

Definition 4.6. Let us define a linear map $\varphi \colon \mathcal{U}_d'^{\leq 0} \to (U_d'^{\geq 0})^*$ by setting

$$\varphi(F_I \sigma_d^m K_\lambda) := f_I k_{m,\lambda}. \tag{4.16}$$

Here we define a bilinear form $(\ |\ ) = (\ |\ )_d \colon U_d'^{\geq 0} \times \mathcal{U}_d'^{\leq 0} \to \mathbb{C}$ by setting

$$(x \,|\, y) := \varphi(y)(x), \tag{4.17}$$

where $x \in U_d'^{\geq 0}$ and $y \in \mathcal{U}_d'^{\leq 0}$. We use the notation $(\ |\ ) = (\ |\ )_d$ also for the bilinear form $(\ |\ ) \colon (U_d'^{\geq 0})^{\otimes 2} \times (\mathcal{U}_d'^{\leq 0})^{\otimes 2} \to \mathbb{C}$ induced by

$$(x_1 \otimes x_2 \,|\, y_1 \otimes y_2) := (x_1 \,|\, y_1)(x_2 \,|\, y_2). \tag{4.18}$$

Lemma 4.7. *Let I, J be arbitrary sequences of elements of Π_d, $m, n \in \{0,1\}$, and $\lambda, \mu \in \frac{1}{2}Q_d$. Then*

$$(E_I \sigma_d^m K_\lambda \,|\, F_J \sigma_d^n K_\mu) = \delta_{\mathrm{wt}(I),\mathrm{wt}(J)} (-1)^{mn} q^{-(\lambda|\mu)} (E_I \,|\, F_J). \tag{4.19}$$

Moreover, for each $x \in U_d'^{\geq 0}$ and $y \in \mathcal{U}_d'^{\leq 0}$, the following equality holds:

$$(x \,|\, y_1 y_2) = (\Delta(x) \,|\, y_1 \otimes y_2). \tag{4.20}$$

Proof. The equality (4.19) follows from Lemma 4.4 and (4.16)(4.17).

We claim that the map φ is an algebra homomorphism. Indeed, by Lemma 4.5 we see that

$$\begin{aligned}
\varphi(F_I \sigma_d^m K_\lambda F_J \sigma_d^n K_\mu) &= (-1)^{mp(\mathrm{wt} J)} q^{-(\lambda|\mathrm{wt} J)} \varphi(F_{(I,J)} \sigma_d^{m+n} K_{\lambda+\mu}) = (-1)^{mp(\mathrm{wt} J)} q^{-(\lambda|\mathrm{wt} J)} f_{(I,J)} k_{m+n,\lambda+\mu} \\
&= (-1)^{mp(\mathrm{wt} J)} q^{-(\lambda|\mathrm{wt} J)} f_I f_J k_{m,\lambda} k_{n,\mu} = f_I k_{m,\lambda} f_J k_{n,\mu} = \varphi(F_I \sigma_d^m K_\lambda) \varphi(F_J \sigma_d^n K_\mu).
\end{aligned}$$

By the claim and (4.17)(4.18), we see that

$$\begin{aligned}
(x \,|\, y_1 y_2) &= \varphi(y_1 y_2)(x) = (\varphi(y_1)\varphi(y_2))(x) = \varphi(y_1) \otimes \varphi(y_2)(\Delta(x)) = \varphi(y_1) \otimes \varphi(y_2)\Big(\sum_i x_i \otimes x_i'\Big) \\
&= \sum_i \varphi(y_1)(x_i) \varphi(y_2)(x_i') = \sum_i (x_i \,|\, y_1)(x_i' \,|\, y_2) = \sum_i (x_i \otimes x_i' \,|\, y_1 \otimes y_2) = (\Delta(x) \,|\, y_1 \otimes y_2).
\end{aligned}$$ □

Lemma 4.8. *Let $\alpha \in \Pi_d$, and $n \in \mathbb{N}$. Then the following equality holds:*

$$(E_\alpha^n \,|\, F_\alpha^n) = q^{n(n-1)/2} [n]_{q_\alpha}! / (q_\alpha^{-1} - q_\alpha)^n. \tag{4.21}$$

Proof. We use the induction on n. In the case where $n = 1$, the claim is clear. Suppose that the formula holds for some $n - 1 \in \mathbb{N}$ with $n \geq 2$. Then, Lemma 4.2 and Lemma 4.7, we see that

$$\begin{aligned}
(E_\alpha^n \,|\, F_\alpha^n) &= (\Delta(E_\alpha^n) \,|\, F_\alpha^{n-1} \otimes F_\alpha) = q_\alpha^{n-1} [n]_{q_\alpha} (E_\alpha^{n-1} \sigma_d^{p(\alpha)} K_\alpha \otimes E_\alpha \,|\, F_\alpha^{n-1} \otimes F_\alpha) \\
&= q_\alpha^{n-1} [n]_{q_\alpha} (E_\alpha^{n-1} \,|\, F_\alpha^{n-1})(E_\alpha \,|\, F_\alpha) = q_\alpha^{n-1} [n]_{q_\alpha} q_\alpha^{(n-1)(n-2)/2} [n-1]_{q_\alpha}! / (q_\alpha^{-1} - q_\alpha)^{n-1} \cdot 1/(q_\alpha^{-1} - q_\alpha) \\
&= q_\alpha^{(n-1)n/2} [n]_{q_\alpha}! / (q_\alpha^{-1} - q_\alpha)^n.
\end{aligned}$$ □

Proposition 4.9. *For all weight vectors $x_1, x_2 \in U_d'^{\geq 0}$ and all $y \in \mathcal{U}_d'^{\leq 0}$, we have the following equality:*

$$(x_1x_2 \,|\, y) = (-1)^{m(x_1)p(\mathrm{wt}x_2)+m(x_2)p(\mathrm{wt}x_1)+p(\mathrm{wt}x_1)p(\mathrm{wt}x_2)}(x_2 \otimes x_1 \,|\, \Delta(y)), \tag{4.22}$$

where $x_r = x_r^+\sigma_d^{m(x_r)}K_{\mu_r}$ with $x_r^+ \in U_d'^{>0}$, $m(x_r) \in \{0,1\}$, and $\mu_r \in \frac{1}{2}Q_d$ for each $r = 1, 2$.

Proof. It suffices to show the equality for weight vectors $y \in \mathcal{U}_d'^{\leq 0}$. We use the induction on $\mathrm{wt}(y)$. In the case where $\mathrm{wt}(y) = 0$, i.e., $y \in \mathcal{U}_d'^0$, thanks to Lemma 4.7, it suffices to consider the case where $x_1, x_2 \in U_d'^0$. Hence, the equality (4.22) is clear in this case. Suppose that $\mathrm{wt}(y) = \alpha$ with $\alpha \in \Pi_d$. Then we may write $y = F_\alpha\sigma_d^lK_\lambda$ with $l \in \{0,1\}$ and $\lambda \in \frac{1}{2}Q_d$. We see that

$$\begin{aligned}
(\sigma_d^mK_\mu \cdot E_\alpha\sigma_d^nK_\nu \,|\, F_\alpha\sigma_d^lK_\lambda) &= (-1)^{mp(\alpha)}q^{(\mu|\alpha)}(E_\alpha\sigma_d^{m+n}K_{\mu+\nu}|F_\alpha\sigma_d^lK_\lambda)\\
&= (-1)^{mp(\alpha)}q^{(\mu|\alpha)}(-1)^{(m+n)l}q^{-(\mu+\nu|\lambda)}/(q_\alpha^{-1}-q_\alpha),\\
(E_\alpha\sigma_d^nK_\nu \otimes \sigma_d^mK_\mu|\Delta(F_\alpha\sigma_d^lK_\lambda)) &= (E_\alpha\sigma_d^nK_\nu \otimes \sigma_d^mK_\mu \,|\, F_\alpha\sigma_d^lK_\lambda \otimes \sigma_d^lK_{\lambda-\alpha} + \sigma_d^{p(\alpha)+l}K_\lambda \otimes F_\alpha\sigma_d^lK_\lambda)\\
&= (-1)^{(m+n)l}q^{-(\mu|\lambda)}q^{-(\mu|\lambda-\alpha)}/(q_\alpha^{-1}-q_\alpha).
\end{aligned}$$

Thus, in the case where $x_1 = \sigma_d^mK_\mu$ and $x_2 = E_\alpha\sigma_d^nK_\nu$, the equality (4.22) is valid. We see that

$$\begin{aligned}
(E_\alpha\sigma_d^nK_\nu \cdot \sigma_d^mK_\mu \,|\, F_\alpha\sigma_d^lK_\lambda) &= (E_\alpha\sigma_d^{m+n}K_{\mu+\nu}|F_\alpha\sigma_d^lK_\lambda) = (-1)^{(m+n)l}q^{-(\mu+\nu|\lambda)}/(q_\alpha^{-1}-q_\alpha),\\
(\sigma_d^mK_\mu \otimes E_\alpha\sigma_d^nK_\nu \,|\, \Delta(F_\alpha\sigma_d^lK_\lambda)) &= (\sigma_d^mK_\mu \otimes E_\alpha\sigma_d^nK_\nu \,|\, F_\alpha\sigma_d^lK_\lambda \otimes \sigma_d^lK_{\lambda-\alpha} + \sigma_d^{p(\alpha)+l}K_\lambda \otimes F_\alpha\sigma_d^lK_\lambda)\\
&= (-1)^{m(p(\alpha)+l)}(-1)^{nl}q^{-(\mu+\nu|\lambda)}/(q_\alpha^{-1}-q_\alpha).
\end{aligned}$$

Thus, in the case where $x_1 = E_\alpha\sigma_d^nK_\nu$ and $x_2 = \sigma_d^mK_\mu$, the equality (4.22) is valid.

We suppose that the equality (4.22) holds for weight vectors $y_1, y_2 \in \mathcal{U}_d'^{\leq 0}$. Firstly, we consider in the case where $x_1, x_2 \in U_d'^{>0}$. Then $m(x_1) = m(x_2) = 0$. For each $r = 1, 2$, we write $\Delta(x_r) = \sum_i x_{ri} \otimes x'_{ri}$ with $x_{ri} \in U_d'^{\geq 0}$ and $x'_{ri} \in U_d'^{>0}$ and $\Delta(y_r) = \sum_i y_{ri} \otimes y'_{ri}$ with $y_{ri}, y'_{ri} \in \mathcal{U}_d'^{\leq 0}$. By Lemma 4.1, we have $m(x_{ri}) = p(\mathrm{wt}(x'_{ri}))$ and $m(x'_{ri}) = 0$. Hence, we see that

$$\begin{aligned}
&(x_1x_2 \,|\, y_1y_2) = (\Delta(x_1x_2) \,|\, y_1 \otimes y_2)\\
&= \sum_{i,j}(x_{1i}x_{2j} \otimes x'_{1i}x'_{2j} \,|\, y_1 \otimes y_2) = \sum_{i,j}(x_{1i}x_{2j} \,|\, y_1)(x'_{1i}x'_{2j} \,|\, y_2)\\
&= \sum_{i,j}(-1)^{p(\mathrm{wt}x'_{1i})p(\mathrm{wt}x_{2j})+p(\mathrm{wt}x'_{2j})p(\mathrm{wt}x_{1i})+p(\mathrm{wt}x_{1i})p(\mathrm{wt}x_{2j})+p(\mathrm{wt}x'_{1i})p(\mathrm{wt}x'_{2j})}(x_{2j} \otimes x_{1i} \,|\, \Delta(y_1))(x'_{2j} \otimes x'_{1i} \,|\, \Delta(y_2))\\
&= \sum_{i,j}(-1)^{p(\mathrm{wt}x_1)p(\mathrm{wt}x_{2j})+p(\mathrm{wt}x_1)p(\mathrm{wt}x'_{2j})}(x_{2j} \otimes x_{1i} \,|\, \Delta(y_1))(x'_{2j} \otimes x'_{1i} \,|\, \Delta(y_2))\\
&= \sum_{i,j}(-1)^{p(\mathrm{wt}x_1)p(\mathrm{wt}x_2)}(x_{2j} \otimes x_{1i} \,|\, \Delta(y_1))(x'_{2j} \otimes x'_{1i} \,|\, \Delta(y_2))\\
&= (-1)^{p(\mathrm{wt}x_1)p(\mathrm{wt}x_2)}\sum_{i,j,k,l}(x_{2j} \otimes x_{1i} \,|\, y_{1k} \otimes y'_{1k})(x'_{2j} \otimes x'_{1i} \,|\, y_{2l} \otimes y'_{2l})\\
&= (-1)^{p(\mathrm{wt}x_1)p(\mathrm{wt}x_2)}\sum_{i,j,k,l}(x_{2j} \,|\, y_{1k})(x_{1i} \,|\, y'_{1k})(x'_{2j} \,|\, y_{2l})(x'_{1i} \,|\, y'_{2l})\\
&= (-1)^{p(\mathrm{wt}x_1)p(\mathrm{wt}x_2)}\sum_{i,j,k,l}(x_{2j} \otimes x'_{2j} \,|\, y_{1k} \otimes y_{2l})(x_{1i} \otimes x'_{1i} \,|\, y'_{1k} \otimes y'_{2l})\\
&= (-1)^{p(\mathrm{wt}x_1)p(\mathrm{wt}x_2)}\sum_{k,l}(\Delta(x_2) \,|\, y_{1k} \otimes y_{2l})(\Delta(x_1) \,|\, y'_{1k} \otimes y'_{2l})\\
&= (-1)^{p(\mathrm{wt}x_1)p(\mathrm{wt}x_2)}\sum_{k,l}(x_2 \,|\, y_{1k}y_{2l})(x_1 \,|\, y'_{1k}y'_{2l})\\
&= (-1)^{p(\mathrm{wt}x_1)p(\mathrm{wt}x_2)}\sum_{k,l}(x_2 \otimes x_1 \,|\, y_{1k}y_{2l} \otimes y'_{1k}y'_{2l}) = (-1)^{p(\mathrm{wt}x_1)p(\mathrm{wt}x_2)}(x_2 \otimes x_1 \,|\, \Delta(y_1y_2)).
\end{aligned}$$

Thus the equality (4.22) is valid in the case where $x_1, x_2 \in U_d'^{>0}$ for all $y \in \mathcal{U}_d'^{\leq 0}$.

Secondly, we prove the equality (4.22) in the general case, i.e., in the case where $x_1, x_2 \in U_d'^{\geq 0}$ and $y \in \mathcal{U}_d'^{\leq 0}$. For each $r = 1,2$ we write $x_r = x_r^+ \sigma_d^{m(x_r)} K_{\mu_r}$ with $x_r^+ \in U_d'^{>0}$, $m(x_r) \in \{0,1\}$, and $\mu_r \in \frac{1}{2}Q_d$, and write $y = y^- \sigma_d^{m(y)} K_\nu$ with $y^- \in \mathcal{U}_d'^{<0}$, $m(y) \in \{0,1\}$, and $\nu \in \frac{1}{2}Q_d$. Then we see that

$$
\begin{aligned}
&(x_1 x_2 \,|\, y) = (x_1^+ \sigma_d^{m(x_1)} K_{\mu_1} x_2^+ \sigma_d^{m(x_2)} K_{\mu_2} \,|\, y^- \sigma_d^{m(y)} K_\nu) \\
&= (-1)^{m(x_1)p(\mathrm{wt}x_2)} q^{(\mu_1|\mathrm{wt}x_2)} (x_1^+ x_2^+ \sigma_d^{m(x_1x_2)} K_{\mu_1+\mu_2} \,|\, y^- \sigma_d^{m(y)} K_\nu) \\
&= (-1)^{m(x_1)p(\mathrm{wt}x_2)} q^{(\mu_1|\mathrm{wt}x_2)} (x_1^+ x_2^+ \,|\, y^-)(-1)^{m(x_1x_2)m(y)} q^{-(\mu_1+\mu_2|\nu)} \\
&= (-1)^{m(x_1)p(\mathrm{wt}x_2)+p(\mathrm{wt}x_1)p(\mathrm{wt}x_2)} q^{(\mu_1|\mathrm{wt}x_2)} (x_2^+ \otimes x_1^+ \,|\, \Delta(y^-))(-1)^{m(x_1x_2)m(y)} q^{-(\mu_1+\mu_2|\nu)} \\
&= (-1)^{m(x_1)p(\mathrm{wt}x_2)+p(\mathrm{wt}x_1)p(\mathrm{wt}x_2)} q^{(\mu_1|\mathrm{wt}x_2)} \sum_k (x_2^+ \otimes x_1^+ \,|\, (y^-)_k^- \otimes (y^-)_k'^-)(-1)^{m(x_1x_2)m(y)} q^{-(\mu_1+\mu_2|\nu)},
\end{aligned}
$$

where

$$
\Delta(y) = \sum_k y_k \otimes y_k' = \Delta(y^- \sigma_d^{m(y)} K_\nu) = \sum_k (y^-)_k^- \sigma_d^{-p(\mathrm{wt}y_k')+m(y)} K_\nu \otimes (y^-)_k'^- \sigma_d^{m(y)} K_{\mathrm{wt}y_k+\nu}
$$

with $(y^-)_k^-$, $(y^-)_k'^- \in \mathcal{U}_d'^{<0}$. On the other hand, we see that

$$
\begin{aligned}
&(x_2 \otimes x_1 \,|\, \Delta(y)) = (x_2^+ \sigma_d^{m(x_2)} K_{\mu_2} \otimes x_1^+ \sigma_d^{m(x_1)} K_{\mu_1} \,|\, \sum_k (y^-)_k^- \sigma_d^{-p(\mathrm{wt}y_k')+m(y)} K_\nu \otimes (y^-)_k'^- \sigma_d^{m(y)} K_{\mathrm{wt}y_k+\nu}) \\
&= \sum_k (x_2^+ \sigma_d^{m(x_2)} K_{\mu_2} \,|\, (y^-)_k^- \sigma_d^{p(\mathrm{wt}x_1)+m(y)} K_\nu)(x_1^+ \sigma_d^{m(x_1)} K_{\mu_1} \,|\, (y^-)_k'^- \sigma_d^{m(y)} K_{-\mathrm{wt}x_2+\nu}) \\
&= \sum_k (-1)^{m(x_2)p(\mathrm{wt}x_1)} q^{(\mu_1|\mathrm{wt}x_2)} (x_2^+ \,|\, (y^-)_k^-)(x_1^+ \,|\, (y^-)_k'^-)(-1)^{m(x_1x_2)m(y)} q^{-(\mu_1+\mu_2|\nu)} \\
&= (-1)^{m(x_2)p(\mathrm{wt}x_1)} q^{(\mu_1|\mathrm{wt}x_2)} \sum_k (x_2^+ \otimes x_1^+ \,|\, (y^-)_k^- \otimes (y^-)_k'^-)(-1)^{m(x_1x_2)m(y)} q^{-(\mu_1+\mu_2|\nu)},
\end{aligned}
$$

where we use $\mathrm{wt}y_k = -\mathrm{wt}(x_2)$ and $\mathrm{wt}(y_k') = -\mathrm{wt}(x_1)$. Therefore, the equality (4.22) is valid for all cases. □

Theorem 4.10. *For each $d \in \mathcal{D}$, there exists a unique bilinear form $(\ |\ ) = (\ |\ )_d \colon U_d'^{\geq 0} \times U_d'^{\leq 0} \to \mathbb{C}$ such that*

$$(x \,|\, y_1 y_2) = (\Delta(x) \,|\, y_1 \otimes y_2), \tag{4.23}$$

$$(E_I \sigma_d^m K_\lambda \cdot E_J \sigma_d^n K_\mu \,|\, y) = (-1)^{mp(\mathrm{wt}J)+np(\mathrm{wt}I)+p(\mathrm{wt}I)p(\mathrm{wt}J)} (E_J \sigma_d^n K_\mu \otimes E_I \sigma_d^m K_\lambda \,|\, \Delta(y)), \tag{4.24}$$

$$(\sigma_d^m K_\lambda \,|\, \sigma_d^n K_\mu) = (-1)^{mn} q^{-(\lambda|\mu)}, \tag{4.25}$$

$$(E_\alpha \,|\, \sigma_d^m K_\lambda) = (\sigma_d^m K_\lambda \,|\, F_\alpha) = 0, \tag{4.26}$$

$$(E_\alpha \,|\, F_\beta) = \delta_{\alpha\beta}/(q_\alpha^{-1} - q_\alpha), \tag{4.27}$$

where $x \in U_d'^{\geq 0}$, $y, y_1, y_2 \in U_d'^{\leq 0}$, I, J are sequences of elements of Π_d, $\lambda, \mu \in \frac{1}{2}Q_d$, $m, n \in \{0,1\}$, and $\alpha, \beta \in \Pi_d$.

Proof. By Lemma 4.7, Lemma 4.8, and Proposition 4.9, we see that there exists a bilinear form $(\ |\ ) = (\ |\ )_d \colon U_d'^{\geq 0} \times \mathcal{U}_d'^{\leq 0} \to \mathbb{C}$ satisfying (4.23)–(4.27) with $y, y_1, y_2 \in \mathcal{U}_d'^{\leq 0}$. Let $\mathcal{I}$ be the two-sided ideal of $\mathcal{U}_d'^{\geq 0}$ generated by the elements displayed in (3.8)–(3.12). Since $U_d'^{\leq 0} = \mathcal{U}_d'^{\leq 0}/\Psi_d(\mathcal{I})$, to prove the existence of the form $(\ |\ ) = (\ |\ )_d$ on $U_d'^{\geq 0} \times U_d'^{\leq 0}$, it suffices to show that for all $x \in U_d'^{\geq 0}$,

$$(x \,|\, \Psi_d(\mathcal{I})) = \{0\}. \tag{4.28}$$

Let $\mathcal{SR}$ be an arbitrary element displayed in (3.8)–(3.11), and set $\mathcal{SR}^- := \Psi_d(\mathcal{SR})$. Then we will show that for all sequences I of elements of Π_d,

$$(E_I \,|\, \mathcal{SR}^-) = 0. \tag{4.29}$$

Here, by Lemma 4.7, we may assume that $\mathrm{wt}(I) = \mathrm{wt}(SR)$. Let us write $I = (\alpha, J)$ with $\alpha \in \Pi_d$ and J a sequence of elements of Π_d. Then, by Proposition 4.9 and Proposition 3.3(1), we see that

$$
\begin{aligned}
(E_I \,|\, \mathcal{SR}^-) &= (-1)^{p(\alpha)p(\mathrm{wt}J)} (E_J \otimes E_\alpha \,|\, \Delta(\mathcal{SR}^-)) = (-1)^{p(\alpha)p(\mathrm{wt}J)} (E_J \otimes E_\alpha \,|\, \mathcal{SR}^- \otimes K_{\mathrm{wt}(I)}^{-1} + \sigma_d^{p(\mathrm{wt}I)} \otimes \mathcal{SR}^-) \\
&= (-1)^{p(\alpha)p(\mathrm{wt}J)} \{(E_J \,|\, SR^-)(E_\alpha \,|\, K_{\mathrm{wt}(I)}^{-1}) + (E_J \,|\, \sigma_d^{p(\mathrm{wt}I)})(E_\alpha \,|\, \mathcal{SR}^-)\}.
\end{aligned}
$$

Since $\mathrm{wt}(J) \neq \mathrm{wt}(\mathcal{SR})$ and $\alpha \neq \mathrm{wt}(\mathcal{SR})$, we have $(E_J \,|\, \mathcal{SR}^-) = (E_\alpha \,|\, \mathcal{SR}^-) = 0$, and hence $(E_I \,|\, \mathcal{SR}^-) = 0$. Let $\mathcal{J}$ be the two-sided ideal of $\mathcal{U}_d'^{\geq 0}$ generated by the elements displayed in (3.8)–(3.11). Then, by (4.2), (4.29), and the property (4.23) of the bilinear form $(\ |\ ) = (\ |\ )_d$ on $U_d'^{\geq 0} \times \mathcal{U}_d'^{\leq 0}$, we see that

$$(E_I \,|\, \Psi_d(\mathcal{J})) = \{0\} \tag{4.30}$$

for all sequences I of elements of Π_d.

Let $\mathcal{SR}(3.12)$ be an arbitrary element displayed in (3.12), and set $\mathcal{SR}^-(3.12) := \Psi_d(\mathcal{SR}(3.12))$. As the argument in the previous paragraph, by Proposition 4.9 and Proposition 3.3(2) with (4.30), we see that $(E_I \,|\, \mathcal{SR}^-(3.12)) = 0$ for all sequences I of elements of Π_d, and hence $(E_I \,|\, \mathcal{U}_d'^{\leq 0} \cdot \mathcal{SR}^-(3.12) \cdot \mathcal{U}_d'^{\leq 0}) = \{0\}$.

By combining the above results, we have shown (4.28). Therefore, there exists a required bilinear form $(\ |\ ) = (\ |\ )_d \colon U_d'^{\geq 0} \times U_d'^{\leq 0} \to \mathbb{C}$. By (4.23) and (4.24), we see that the values displayed in (4.25), (4.26), and (4.27) determine the values on the whole algebras, which implies the uniqueness. □

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