Z-transform of Unit Pulse Trains using Signal processing

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信号処理によるユニット・パルス列のZ変換

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This paper are presented the signal processing by Z-transformation on the signal as the Unit Pulse Train of unipolar, or bipolar seqences. And the method is effectively the operation to a discrete-time signals: periodic pulse trains.

Up to the present day, the signal processing method of the circuits for informative processing has desired to apper a new method and to develop a algorism for it, because these calculation is in need of too many calculated time and complicacy. In this point, it is considered to apply the method of Z-transform for the output signal of output circuits. In this paper, it is reported to consider for a few example in the case of Unit Pulse Train (i.e P.S.N.)*1 generated the output as sampler signal, so that this method may be effectived more than the another.

(I) Z-transformation procedure as Signal processing

Generaly, the output of sampler criuits have be given by a input signal $f_S(t)$ multiplied by sampling signal $p(t, \tau)$.

In symbols, $f^*(t) = p(t, \tau) \cdot f_s(t)$ ··············①

The proximate unit pulse trains with the duration time τ of $p(t, \tau)$ assumed as $\tau \to o$, can be represented as follows:

$$P(t) = \sum_{n=o}^{+\infty} \delta_T (t-nT). \quad \cdots\cdots ②^{*2}$$

Hence, the output of sampler circuits is given by the following equation③.

$$f^*(t) = \sum_{n=o}^{+\infty} \delta_T (t-nT) \cdot f_s(t) \cdots\cdots ③^{*3}$$

Taking the Laplace transform of Equation ③, it obtain

$$F^*(s) = \sum_{n=o}^{+\infty} f_s(nT) \cdot e^{-nsT} \cdots\cdots ④$$

Furthermore taking the Z-transform of Equation ④, Equation ①is given to transform Z-transformation.

$$\text{Thus, } F(z) = F^*(s) \Big|_{s=\frac{1}{T}\ln z} = \sum_{n=o}^{+\infty} f_s(nT) \cdot z^{-n} = f_o(o) \cdot z^{o} + f_s(T) \cdot z^{-1} + f_s(2T) \cdot z^{-2} + \cdots. \quad \cdots\cdots ⑤$$

Equation ⑤ replaced the discrete signal means to represent the sampled values shifting by every T and to form a polynominal which the first term $f_S(o) \cdot z^{o}$ find out the initial sampled value and the second term find out the value of more shifting by T from the first term, the next term have the same shifting value, as before.

After the resulted Z-transform, these output of sampler circuit become a pulse train, furthermore, the pulse trains have be normalized for the unit pulse trains by weighting function (for example, reciprocal of input signal: $1/f_S(t)$) .

(Ⅱ) **Calculated examples on Unit pulse trains of Unipolar sequence**

When the output of sampler circuits shown in Figure I are generated N number within section T_S of the isolated group on unit pulse, it consider only the first periodic pulse trains that is the group of continous unit pulses sequence.

The output are represened as follows by Equation ⑥

Thus, $$F_1(z) = \sum_{k=o}^{N-1} f\,(k) \cdot z^{-k} \cdots\cdots\cdots\cdots ⑥^{*4}$$

For example, in the case of the first periodic group of pulse train is N= 3 shown in Figure 1, $F_1(z)$ can be calculated by Eq. ⑥ .

$$F_1(z) = \sum_{k=o}^{2} 1 \cdot z^{-k} + \sum_{k=3}^{5} 0 \cdot z^{-k} = \frac{1+z+z^2}{z^2}$$

Next consideration is about the continuous pulsed group which the above isolated group generate to repeate at the m-th numbers and then this equation are given by

$$\sum_{m=o}^{\infty} z^{-mN} = \sum_{m=o}^{\infty} (z^{-N})^m = \frac{z^N}{z^N-1} \cdots\cdots ⑦$$

for $|z| > 1$.

Hence, the periodic data signal as sampled values shown in Fig. 1 may be represented by F(z) as follows.

$$F(z) = \left(\frac{z^N}{z^N-1}\right) \cdot F_1(z) \cdots\cdots\cdots\cdots ⑧$$

For above example, the resultant equation is

$$F(z) = \left(\frac{z^6}{z^6-1}\right) \cdot F_1(z) = \frac{z^4}{(z-1)(z^3+1)}$$

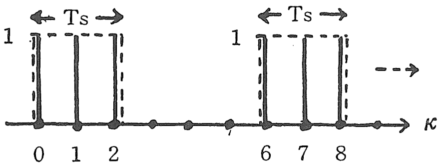

Figure 1 Unipolar Sequence

(Ⅲ) **Caluculated examples on Unit pulse trains of Bipolar sequence**

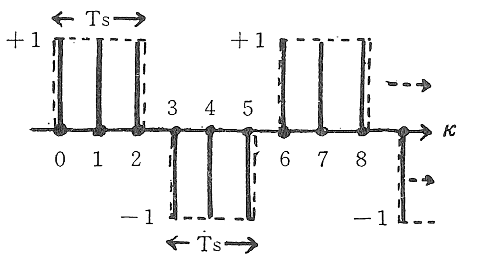

Figure 2 Bipolar Sequence

As shown in Figure 2, the group in continuous unit pulses whithin one period can be represented as follow, by the calculation of identical procedure as (Ⅱ).

$$F_1(z) = \sum_{k=o}^{2} 1 \cdot z^{-k} + \sum_{k=3}^{5} (-1) \cdot z^{-k} = \frac{(z^3-1)\ (z^2+z+1)}{z^5}$$

In the result, an equation of Bipolar sequence is

$$F(z) = \frac{z\,(z^2+z+1)}{z^3+1}$$

Conclusion

The following became clear after the discussion on the above results. To apply the Z-transformation as signal processing of P. S. N. may be effectived not only making up the simple algebraic manipulation but also by easy calculation.

Reference

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Apendix

*1 P. S. N.………Abbreviation of Periodic Sequential Number

*2 Derivation of Equation ②

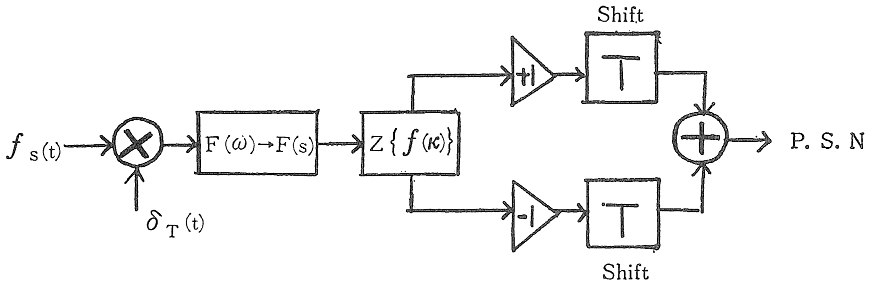

Figure 3
Block diagram of Signal processing to obtain P. S. N

When p(t, τ) form a rectuangular pulse, the Fourier series representation can be obtain as,

$$p(t,\tau) = \sum_{n=-\infty}^{\infty} F_n e^{-jn\omega_o t}$$

where Fourier coefficients are

$$F_n = \frac{V}{T}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} e^{-jn\omega_o t}\,dt = \frac{V\cdot\tau}{T}\ \mathrm{sinc}\left(\frac{n\omega_o t}{2}\right)$$

Next, taking the limit of F_n,

$$\lim_{\substack{V\cdot\tau\to 1\\ \tau\to 0}} F_n = \frac{1}{T}$$

Thus, p(t, τ) becoming the unit pulse train is taken the Fourier transform. These equation become

$$P(\omega) = F\left\{\frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\omega_o t}\right\} = \frac{2\pi}{T}\sum_{n=-\infty}^{\infty}\delta_T(\omega - n\omega_o)$$

and also, in the time domain it obtain

$$P\ (t) = \sum_{n=-\infty}^{\infty} \delta_T\ (t-nT)$$

And otherwise, the limit of summation can be changed as follows, because $P(\omega)$ are the even function of $n\omega_o$ and n is positive integer or $t>0$. Thus,

$$P(t) = \sum_{n=o}^{\infty} \delta_T\ (t-nT)$$

*3 Derivation of Eq. ④ from Eq. ③

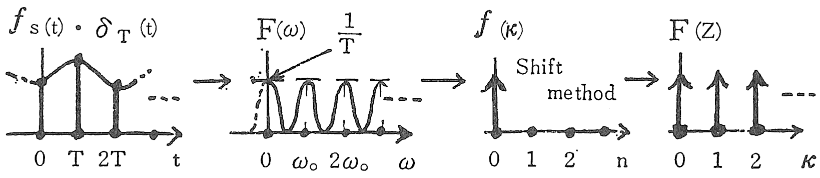

Figure 4
Rolationship of Signal Process

$$\mathcal{L}\{f^*(t)\} = \sum_{n=o}^{\infty} f_s\ (nT)\ \mathcal{L}\{\delta_T\ (t-nT)\} = \sum_{n=o}^{\infty} f_s(nT)e^{-jn\omega T} = \sum_{n=o}^{\infty} f_s(nT)e^{-nsT}$$

*4 Illustration of Eq. ⑤

At Eq. ⑥ , in the case multiplied $f_S(nT)$ by weighting function $\mathcal{W}(t)$, replacing with $k = nT$ and $T = 1$ the sequence denote $f(k)$, it is given by

$$f\ (k) = \begin{cases} 0 & \text{for } k = -1,\ -2,\cdots\cdots \\ 1 & \text{for } k = 0,\ 1,\ 2,\cdots\cdots \end{cases}$$

A sequence of number is $\{f(k)\}$ which satisfy the property as follows, $f(k)=f(k+N)$

and generaly, $Z\{f(k)\} = F(z)$. ···Eq. ⑥.

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